Chapter 4 — Integral calculus and numerical methods

Exercise 4A — Approximation using the derivative

1 a Show that, for small x_1, $3\sin(x) \approx 3x$

$$\begin{aligned} f(x_0+h) &\approx f(x_0) + hf'(x_0) \\ \text{Let } f(x) &= 3\sin(t) \\ \text{So } f'(x) &= 3\cos(t) \\ \text{Let } x_0 &= 0 \text{ and } h = x \\ f(x) &\approx f(0) + x - f'(0) \\ \sin(x) &\approx 3\sin(0) + x \times 3\cos(0) \\ \sin(x) &\approx 0 + x \times 3 \\ \sin(x) &\approx 3x \end{aligned}$$

b $3\sin\left(\frac{\pi}{6}\right) \approx 3\times\frac{\pi}{6}$

$3\sin\left(\frac{\pi}{6}\right) \approx \frac{\pi}{2}$

c Percentage error $= \dfrac{\left|3\sin\frac{\pi}{6} - \frac{\pi}{2}\right|}{3\sin\frac{\pi}{6}} \times 100\%$

$\approx 4.7\%$

2 a Find an approximation for $f(x) = x\cos(x)$ for small values of x

$$\begin{aligned} f(x_0+h) &\approx f(x_0) + hf'(x_0) \\ f(x) &= x\cos(x) \\ f'(x) &= \cos(x) - x\sin(x) \\ \text{Let } x_0 &= 0 \text{ and } h = x \\ f(x) &\approx f(0) + xf'(0) \\ x\cos(x) &\approx 0\times\cos(0) + x(\cos(0) - 0\times\sin(0)) \\ x\cos(x) &\approx 0 + x(1-0) \\ x\cos(x) &\approx x \end{aligned}$$

b $\frac{\pi}{8}\cos\left(\frac{\pi}{8}\right) \approx \frac{\pi}{8}$

3 a Find an approximation for $f(x) = (x+1)^3$ for small values of x

$$\begin{aligned} f(x_0+h) &\approx f(x_0) + hf'(x_0) \\ f(x) &= (x+1)^3 \\ f'(x) &= 3(x+1)^2 \\ \text{Let } x_0 &= 0 \text{ and } h = x \\ f(x) &\approx f(0) + xf'(0) \\ (x+1)^3 &\approx (0+1)^3 + x\times 3(0+1)^2 \\ (x+1)^3 &\approx 1 + x\times 3 \\ (x+1)^3 &\approx 3x+1 \end{aligned}$$

b $1.09^3 = (1+0.9)^3$

$1.09^3 \approx 3\times 0.9 + 1$

$1.09^3 \approx 1.27$

4 Estimate value of $\sqrt{84}$

$$\begin{aligned} f(x_0+h) &\approx f(x_0) + hf'(x_0) \\ \text{Let } f(x) &= \sqrt{x} = x^{\frac{1}{2}} \\ \text{So } f'(x) &= \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}} \\ \text{Let } x_0 &= 81 \text{ and } h = 3 \\ f(81+3) &\approx f(81) + 3f'(81) \\ \sqrt{84} &\approx \sqrt{81} + 3\times\frac{1}{2\sqrt{81}} \\ \sqrt{84} &\approx 9 + \frac{3}{18} \\ \sqrt{84} &\approx 9\frac{1}{6} \end{aligned}$$

5 a $\sqrt{96}$

$$\begin{aligned} f(x_0+h) &\approx f(x_0) + hf'(x_0) \\ \text{Let } f(x) &= \sqrt{x} = x^{\frac{1}{2}} \\ \text{So } f'(x) &= \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}} \\ \text{Let } x_0 &= 100 \text{ and } h = -4 \\ f(100-4) &\approx f(100) - 4f'(100) \\ \sqrt{96} &\approx \sqrt{100} - 4\times\frac{1}{2\sqrt{100}} \\ \sqrt{96} &\approx 10 - \frac{2}{10} \\ \sqrt{96} &\approx 9.8 \end{aligned}$$

b $\frac{1}{0.98}$

$$\begin{aligned} f(x_0+h) &\approx f(x_0) + hf'(x_0) \\ \text{Let } f(x) &= \frac{1}{x} = x^{-1} \\ \text{So } f'(x) &= -x^{-2} = \frac{-1}{x^2} \\ \text{Let } x_0 &= 1 \text{ and } h = -0.02 \\ f(1-0.02) &\approx f(1) - 0.02f'(1) \\ \frac{1}{0.98} &\approx \frac{1}{1} - 0.02\times\left(\frac{-1}{1^2}\right) \\ \frac{1}{0.98} &\approx 1 + 0.02 \\ \frac{1}{0.98} &\approx 1.02 \end{aligned}$$

c $\sqrt{26}$

$$\begin{aligned} f(x_0+h) &\approx f(x_0) + hf'(x_0) \\ \text{Let } f(x) &= \sqrt{x} = x^{\frac{-1}{2}} \\ \text{so } f'(x) &= \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}} \\ \text{Let } x_0 &= 25 \text{ and } h = 1 \\ f(25+1) &\approx f(25) + 1\times f'(25) \\ \sqrt{26} &\approx \sqrt{25} + \frac{1}{2\sqrt{25}} \\ \sqrt{26} &\approx 5 + \frac{1}{2\times 5} \\ \sqrt{26} &\approx 5.1 \end{aligned}$$

d $\sqrt[3]{28}$

$$\begin{aligned} f(x_0+h) &\approx f(x_0) + hf'(x_0) \\ \text{Let } f(x) &= \sqrt[3]{x} = x^{\frac{1}{3}} \\ \text{So } f'(x) &= \frac{1}{3}x^{\frac{-2}{3}} = \frac{1}{3\sqrt[3]{x^2}} = \frac{1}{3(\sqrt[3]{x})^2} \\ \text{Let } x_0 &= 27 \text{ and } h = 1 \\ f(27+1) &\approx f(27) + 1\times f'(27) \\ \sqrt[3]{28} &\approx \sqrt[3]{27} + \frac{1}{3(\sqrt[3]{27})^2} \\ \sqrt[3]{28} &\approx 3 + \frac{1}{3\times 3^2} \\ \sqrt[3]{28} &\approx 3\frac{1}{27} \end{aligned}$$

6 a Let $f(r) = \frac{4}{3}\pi r^3$, so $f'(r) = 4\pi r^2$

Let $v_1 = \frac{4}{3}\pi r^3 = f(r)$ be the initial volume and $v_2 = \frac{4}{3}\pi(r+\Delta r)^3 = f(r+\Delta r)$ be the final volume.

So $v_2 - v_1 = \Delta v$ is the change in volume

$f(x_0 + h) \approx f(x_0) + hf'(x_0)$

Let $x_0 = r$ and $h = \Delta r$

$f(r+\Delta r) \approx f(r) + \Delta r\, f'(r)$

$v_2 \approx v_1 + \Delta r \times 4\pi r^2$

$v_2 - v_1 \approx 4\pi r^2 \Delta r$

$\Delta v \approx 4\pi r^2 \Delta r$, as required.

b Find Δv when the radius changes from 8 cm to 8.2 cm. Let $r = 8$ cm be the initial radius and $\Delta r = 8.2 - 8 = 0.2$ cm be the change in radius.

$\Delta v = 4\pi r^2\, \Delta r$

$\Delta v = 4\pi \times 8^2 \times 0.2$

$\Delta v \approx 160.8\text{ cm}^3$

7 The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$

Let $f(r) = \frac{4}{3}\pi r^3$, so $f'(r) = 4\pi r^2$

Let $V_1 = \frac{4}{3}\pi r^3 = f(r)$ be the initial volume and $V_2 = \frac{4}{3}\pi(r+\Delta r)^3 = f(r+\Delta r) = 1.03V_1$ be the final volume.

$f(x_0 + h) \approx f(x_0) + hf'(x_0)$

Let $x_0 = r$ and $h = \Delta r$

$f(r+\Delta r) \approx f(r) + \Delta r\, f'(r)$

$1.03V_1 \approx V_1 + \Delta r \times 4\pi r^2$

$0.03V_1 \approx 4\pi r^2\, \Delta r$

$0.03 \times \frac{4}{3}\pi r^3 \approx 4\pi r^2 \Delta r$

$0.03 \times \frac{\not{4}}{3}\not{\pi} r^{\not{3}} \approx \not{4}\not{\pi}\not{r^2}\Delta r$

$0.01r \approx \Delta r$

$\Delta r \approx 0.01r$

So a 3% increase in volume gives a 1% increase in radius.

8 The surface area of a sphere is given by $A = 4\pi r^2$

Let $f(r) = 4\pi r^2$, so $f'(r) = 8\pi r$

Let $A_1 = 4\pi r^2 = f(r)$ be the initial surface area and $A_2 = 4\pi(r+\Delta r)^2 = f(r+\Delta r) = 1.03A$, be the final surface area.

$f(x_0 + h) \approx f(x_0) + hf'(x_0)$

Let $x_0 = r$ and $h = \Delta r$

$f(r+\Delta r) \approx f(r) + hf'(\Delta r)$

$1.03A_1 \approx A_1 + \Delta r \times 8\pi r$

$0.03A_1 \approx 8\pi r\, \Delta r$

$0.03 \times 4\pi r^2 \approx 8\pi r\, \Delta r$

$0.03 \times 4\not{\pi} r^{\not{2}} \approx 8\not{\pi}\not{r}\Delta r$

$0.03r \approx 2\, \Delta r$

$\Delta r \approx 0.015r$

So a 3% increase in volume gives a 1.5% increase in radius.

Investigation — Polynomial approximations

1 $f(x) = \sin(x)$, $x_0 = 0$

$$f(x) = f(x_0) + f'(x_0)x + f^{(2)}(x_0)\frac{x^2}{2!} + f^{(3)}(x_0)\frac{x^3}{3!} + \ldots$$

$$= f(0) + f'(0)x + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(x_0)\frac{x^3}{3!} + \ldots$$

$f(x) = \sin(x)$ $\quad f(0) = \sin(0) = 0$

$f'(x) = \cos(x)$ $\quad f'(0) = \cos(0) = 1$

$f^{(2)}(x) = -\sin(x)$ $\quad f^{(2)}(0) = -\sin(0) = 0$

$f^{(3)}(x) = -\cos(x)$ $\quad f^{(3)}(0) = -\cos(0) = -1$

$f^{(4)}(x) = \sin(x)$ $\quad f^{(4)}(0) = \sin(0) = 0$

$f^{(5)}(x) = \cos(x)$ $\quad f^{(5)}(0) = \cos(0) = 1$

$f^{(6)}(x) = -\sin(x)$ $\quad f^{(6)}(0) = \sin(0) = 0$

$f^{(7)}(x) = -\cos(x)$ $\quad f^{(7)}(0) = -\cos(0) = 1$

So $f(x) = 0 + 1 \times x + 0 \times \frac{x^2}{2!} - 1 \times \frac{x^3}{3!} + 0 \times \frac{x^4}{4!} + 1 \times \frac{x^5}{5!} + 0 \times \frac{x^6}{6!} - 1 \times \frac{x^7}{7!}$

$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$, as required.

2 a $f(x) = \cos(x)$

$f(x) = f(0) + f'(0)x + f^{(2)}(x)\frac{x^2}{2!} + f^{(3)}(x)\frac{x^3}{3!} + \ldots$

$f(x) = \cos(x)$ $\quad f(0) = \cos(0) = 1$
$f'(x) = -\sin(x)$ $\quad f'(0) = -\sin(0) = 0$
$f^{(2)}(x) = -\cos(x)$ $\quad f^{(2)}(0) = -\cos(0) = -1$
$f^{(3)}(x) = \sin(x)$ $\quad f^{(3)}(0) = \sin(0) = 0$
$f^{(4)}(x) = \cos(x)$ $\quad f^{(4)}(0) = \cos(0) = 1$ etc.

So $f(x) = 1 + 0 \times x - 1 \times \frac{x^2}{2!} + 0 \times \frac{x^3}{3!} + 1 \times \frac{x^4}{4!} + 0 \times \frac{x^5}{5!} - 1 \times \frac{x^6}{6!} + \ldots$

$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

b $f(x) = e^x$

$f(x) = f(0) + f'(0)x + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \ldots$

$f(x) = e^x$ $\quad f(0) = e^0 = 1$
$f'(x) = e^x$ $\quad f'(0) = e^0 = 1$ etc.

So $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

c $f(x) = \ln(1 + x)$

$f(x) = f(0) + f'(0)x + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \ldots$

$f(x) = \ln(1 + x)$ $\quad f(0) = \ln(1) = 0$
$f'(x) = \frac{1}{1+x} = (1 + x)^{-1}$ $\quad f'(0) = \frac{1}{1} = 1$
$f^{(2)}(x) = -(1 + x)^{-2}$ $\quad f^{(2)}(0) = -(1)^{-2} = -1 = -1!$
$f^{(3)}(x) = 2(1 + x)^{-3}$ $\quad f^{(3)}(0) = 2(1)^{-3} = 2 = 2!$
$f^{(4)}(x) = -3!(1 + x)^{-4}$ $\quad f^{(4)}(0) = -3!(1)^{-4} = -6 = -3!$
$f^{(5)}(x) = 4!(1 + x)^{-5}$ $\quad f^{(5)}(0) = 4!(1 + x)^{-5} = 24 = 4!$ etc.

So $f(x) = 0 + 1 \times x - \frac{1!}{2!}x^2 + \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4 + \ldots$

$f(x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots$

3 a $\sin(2) \approx 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!}$

$\sin(2) \approx 0.91$ (to 2 decimal places)

b $\cos(4.5) \approx 1 - \frac{4.5^2}{2!} + \frac{4.5^4}{4!} - \frac{4.5^6}{6!} + \frac{4.5^8}{8!} - \frac{4.5^{10}}{10!} + \frac{4.5^{12}}{12!} - \frac{4.5^{14}}{14!} + \frac{4.5^{16}}{16!}$

$\cos(4.5) \approx -0.211$ (to 3 decimal places)

c $e^3 \approx 1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \frac{3^5}{5!} + \frac{3^6}{6!} + \frac{3^7}{7!} + \frac{3^8}{8!} + \frac{3^9}{9!} + \frac{3^{10}}{10!} + \frac{3^{11}}{11!} + \frac{3^{12}}{12!} + \frac{3^{13}}{13!}$

$e^3 \approx 20.0855$ (to 4 decimal places)

d $\ln(1.5) = \ln(1 + 0.5)$

$\ln(1.5) \approx 0.5 - \frac{0.5^2}{2} + \frac{0.5^3}{3} - \frac{0.5^4}{4} + \frac{0.5^5}{5} - \frac{0.5^6}{6} + \frac{0.5^7}{7} - \frac{0.5^8}{8}$

$\ln(1.5) \approx 0.405$ (to 3 decimal places)

Exercise 4B — Substitution where the derivative is present in the integrand

1 a $\int 2x(x^2 + 3)^4\,dx$

Let $u = x^2 + 3$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

So $\int 2x(x^2 + 3)^4\,dx$

$= \int 2x(u)^4 \frac{du}{2x}$

$= \int u^4 \, du$

$= \frac{1}{5} u^5 + c$

$= \frac{1}{5} (x^2 + 3)^5 + c$

b $\int 2x(6 - x^2)^{-3} \, dx$

Let $u = 6 - x^2$

$\frac{du}{dx} = -2x$ or $dx = \frac{du}{-2x}$

So $\int 2x(6 - x^2)^{-3} \, dx$

$= \int 2x(u)^{-3} \frac{du}{-2x}$

$= -\int u^{-3} \, du$

$= -\frac{1}{-2} u^{-2} + c$

$= \frac{1}{2u^2} + c$

$= \frac{1}{2(6 - x^2)^2} + c$

c $\int 3x^2(x^3 - 2)^5 \, dx$

Let $u = x^3 - 2$

$\frac{du}{dx} = 3x^2$ or $dx = \frac{du}{3x^2}$

So $\int 3x^2(x^3 - 2)^5 \, dx$

$= \int 3x^2(u)^5 \frac{du}{3x^2}$

$= \int u^5 \, du$

$= \frac{1}{6} u^6 + c$

$= \frac{(x^3 - 2)^6}{6} + c$

d $\int 2(x + 2)(x^2 + 4x)^{-3} \, dx$

Let $u = x^2 + 4x$

$\frac{du}{dx} = 2x + 4$ or $dx = \frac{du}{2x + 4}$

So $\int 2(x + 2)(x^2 + 4x)^{-3} \, dx$

$= \int (2x + 4)(x^2 + 4x)^{-3} \, dx$

$= \int (2x + 4)(u)^{-3} \frac{du}{2x + 4}$

$= \int u^{-3} \, du$

$= \frac{1}{-2} u^{-2} + c$

$= \frac{-1}{2u^2} + c$

$= \frac{-1}{2(x^2 + 4x)^2} + c$

e $\int (2x + 5)\sqrt{x^2 + 5x} \, dx$

Let $u = x^2 + 5x$

$\frac{du}{dx} = 2x + 5$ or $dx = \frac{du}{2x + 5}$

So $\int (2x + 5)\sqrt{x^2 + 5x} \, dx$

$= \int (2x + 5)(x^2 + 5x)^{\frac{1}{2}} \, dx$

$= \int (2x + 5)(u)^{\frac{1}{2}} \frac{du}{2x + 5}$

$= \int u^{\frac{1}{2}} \, du$

$= \frac{2}{3} u^{\frac{3}{2}} + c$

$= \frac{2(x^2 + 5x)^{\frac{3}{2}}}{3} + c$

f $\int \frac{2x - 3}{(x^2 - 3x)^4} \, dx$

Let $u = x^2 - 3x$

$\frac{du}{dx} = 2x - 3$ or $dx = \frac{du}{2x - 3}$

So $\int \frac{2x - 3}{(x^2 - 3x)^4} \, dx$

$= \int \frac{2x - 3}{(u)^4} \times \frac{du}{2x - 3}$

$= \int u^{-4} \, du$

$= \frac{1}{-3} u^{-3} + c$

$= \frac{-1}{3u^3} + c$

$= \frac{-1}{3(x^2 - 3x)^3} + c$

g $\int 3x^2(x^3 - 5)^2 \, dx$

Let $u = x^3 - 5$

$\frac{du}{dx} = 3x^2$ or $dx = \frac{du}{3x^2}$

So $\int 3x^2(x^3 - 5)^2 \, dx$

$= \int 3x^2(u)^2 \frac{du}{3x^2}$

$= \int u^2 \, du$

$= \frac{1}{3} u^3 + c$

$= \frac{(x^3 - 5)^3}{3} + c$

h $\int \frac{3x^2 + 4x}{\sqrt{x^3 + 2x^2}} \, dx$

Let $u = x^3 + 2x^2$

$\frac{du}{dx} = 3x^2 + 4x$ or $dx = \frac{du}{3x^2 + 4x}$

So $\int \frac{3x^2 + 4x}{\sqrt{x^3 + 2x^2}} \, dx$

$= \int \frac{3x^2 + 4x}{\sqrt{u}} \times \frac{du}{3x^2 + 4x}$

$= \int u^{\frac{-1}{2}} \, du$

$= 2u^{\frac{1}{2}} + c$

$= 2(x^3 + 2x^2)^{\frac{1}{2}} + c$

$= 2\sqrt{x^3 + 2x^2} + c$

i $\int 4x^3 e^{x^4} \, dx$

Let $u = x^4$

$\frac{du}{dx} = 4x^3$ or $dx = \frac{du}{4x^3}$

$\int 4x^3 e^{x^4}\, dx$

$= \int 4x^3 e^u \frac{du}{4x^3}$

$= \int e^u\, du$

$= e^u + c$

$= e^{x^4} + c$

j $\int (2x+3)\sin(x^2+3x-2)\,dx$

Let $u = x^2 + 3x - 2$

$\frac{du}{dx} = 2x + 3$ or $dx = \frac{du}{2x+3}$

So $\int (2x+3)\sin(x^2+3x-2)\,dx$

$= \int (2x+3)\sin(u)\frac{du}{2x+3}$

$= \int \sin(u)\,du$

$= -\cos(u) + c$

$= -\cos(x^2 + 3x - 2) + c$

k $\int (3x^2+5)\cos(x^3+5x)\,dx$

Let $u = x^3 + 5x$

$\frac{du}{dx} = 3x^2 + 5$ or $dx = \frac{du}{3x^2+5}$

So $\int (3x^2+5)\cos(x^3+5x)\,dx$

$= \int (3x^2+5)\cos(u)\frac{du}{3x^2+5}$

$= \int \cos(u)\,du$

$= \sin(u) + c$

$= \sin(x^3 + 5x) + c$

l $\int \cos(x)\sin^3(x)\,dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

So $\int \cos(x)\sin^3(x)\,dx$

$= \int \cos(x)(u)^3 \frac{du}{\cos(x)}$

$= \int u^3\,du$

$= \frac{1}{4}u^4 + c$

$= \frac{\sin^4(x)}{4} + c$

m $\int -\sin^4(x)\cos(x)\,dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

So $\int -\sin^4(x)\cos(x)\,dx$

$= \int -(u)^4\cos(x)\frac{du}{\cos(x)}$

$= -\int u^4\,du$

$= -\frac{1}{5}u^5 + c$

$= -\frac{\sin^5(x)}{5} + c$

n $\int \frac{\log_e(x)}{x}\,dx$

Let $u = \log_e(x)$

$\frac{du}{dx} = \frac{1}{x}$ or $dx = x\,du$

So $\int \frac{\log_e(x)}{x}\,dx$

$= \int \frac{u}{x} x\,du$

$= \int u\,du$

$= \frac{1}{2}u^2 + c$

$= \frac{(\log_e(x))^2}{2} + c$

o $\int \sec^2(x)\tan^3(x)\,dx$

Let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$ or $dx = \frac{du}{\sec^2(x)}$

So $\int \sec^2(x)\tan^3(x)\,dx$

$= \int \sec^2(x)(u)^3\frac{du}{\sec^2(x)}$

$= \int u^3\,du$

$= \frac{1}{4}u^4 + c$

$= \frac{\tan^4(x)}{4} + c$

p $\int \frac{(\sin^{-1}(x))^2}{\sqrt{1-x^2}}\,dx$

Let $u = \sin^{-1}(x)$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ or $dx = \sqrt{1-x^2}\,du$

So $\int \frac{(\sin^{-1}(x))^2}{\sqrt{1-x^2}}\,dx$

$= \int \frac{(u)^2}{\sqrt{1-x^2}}\sqrt{1-x^2}\,du$

$= \int u^2\,du$

$= \frac{1}{3}u^3 + c$

$= \frac{(\sin^{-1}(x))^3}{3} + c$

2 Given $\frac{d}{dx}\left[(x^2+5x)^4\right] = 4(2x+5)(x^2+5x)^3$

So $(x^2+5x)^4 + c = \int 4(2x+5)(x^2+5x)^3\,dx$

$2(x^2+5x)^4 + c = 2\int 4(2x+5)(x^2+5x)^3\,dx$

$2(x^2+5x)^4 + c = \int 8(2x+5)(x^2+5x)^3\,dx$

$\int 8(2x+5)(x^2+5x)^3\,dx = 2(x^2+5x)^4 + c$

$\therefore$ **A**

3 a D

b $\int \frac{x}{\sqrt{x^2+3}}\,dx$

Let $u = x^2 + 3$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

So $\int \frac{x}{\sqrt{x^2+3}}\,dx$

$= \int \frac{x}{\sqrt{u}} \frac{du}{2x}$

$= \frac{1}{2}\int u^{\frac{-1}{2}}\,du$

$\therefore$ **B**

c $\int \frac{x}{\sqrt{x^2+3}}\,dx$

$= \frac{1}{2}\int u^{\frac{-1}{2}}\,du,\ u = x^2+3$

$= \frac{1}{2}(2u^{\frac{1}{2}}) + c$

$= u^{\frac{1}{2}} + c$

$= (x^2+3)^{\frac{1}{2}} + c$

$\therefore$ **E**

4 a $\int 6x^2(x^3-2)^5\,dx$

Let $u = x^3 - 2$

$\frac{du}{dx} = 3x^2$ or $dx = \frac{du}{3x^2}$

So $\int 6x^2(x^3-2)^5\,dx$

$= \int 6x^2(u)^5 \frac{du}{3x^2}$

$= 2\int u^5\,du$

$= 2\left(\frac{1}{6}u^6\right) + c$

$= \frac{1}{3}u^6 + c$

$= \frac{(x^3-2)^6}{3} + c$

b $\int x(4-x^2)^3\,dx$

Let $u = 4 - x^2$

$\frac{du}{dx} = -2x$ or $dx = \frac{du}{-2x}$

So $\int x(4-x^2)^3\,dx$

$= \int x(u)^3 \frac{du}{-2x}$

$= \frac{-1}{2}\int u^3\,du$

$= \frac{-1}{2}\left(\frac{1}{4}u^4\right) + c$

$= \frac{-1}{8}u^4 + c$

$= \frac{-(4-x^2)^4}{8} + c$

c $\int x^2(x^3-1)^7\,dx$

Let $u = x^3 - 1$

$\frac{du}{dx} = 3x^2$ or $dx = \frac{du}{3x^2}$

So $\int x^2(x^3-1)^7\,dx$

$= \int x^2(u)^7 \frac{du}{3x^2}$

$= \frac{1}{3}\int u^7\,du$

$= \frac{1}{3}\left(\frac{1}{8}u^8\right) + c$

$= \frac{4^8}{24} + c$

$= \frac{(x^3-1)^8}{24} + c$

d $\int (x+3)(x^2+6x-2)^4\,dx$

Let $u = x^2 + 6x - 2$

$\frac{du}{dx} = 2x+6$ or $dx = \frac{du}{2x+6}$

So $\int (x+3)(x^2+6x-2)^4\,dx$

$= \int (x+3)(u)^4 \frac{du}{2x+6}$

$= \int (x+3)u^4 \frac{du}{2(x+3)}$

$= \frac{1}{2}\int u^4\,du$

$= \frac{1}{2}\left(\frac{1}{5}u^5\right) + c$

$= \frac{u^5}{10} + c$

$= \frac{(x^2+6x-2)^5}{10} + c$

e $\int (x+1)(x^2+2x+3)^{-4}\,dx$

Let $u = x^2 + 2x + 3$

$\frac{du}{dx} = 2x+2$ or $dx = \frac{du}{2x+2}$

So $\int (x+1)(x^2+2x+3)^{-4}\,dx$

$= \int (x+1)(u)^{-4} \frac{du}{2x+2}$

$= \int (x+1)u^{-4} \frac{du}{2(x+1)}$

$= \frac{1}{2}\int u^{-4}\,du$

$= \frac{1}{2}\left(\frac{1}{-3}u^{-3}\right) + c$

$= \frac{-u^{-3}}{6} + c$

$= \frac{-(x^2+2x+3)^{-3}}{6} + c$

f $\int \frac{4x+6}{\sqrt{x^2+3x}}\,dx$

Let $u = x^2 + 3x$

$\frac{du}{dx} = 2x+3$ or $dx = \frac{du}{2x+3}$

So $\int \frac{4x+6}{\sqrt{x^2+3x}}\,dx$

$= \int \frac{2(2x+3)}{\sqrt{u}} \frac{du}{2x+3}$

$= 2\int u^{\frac{-1}{2}}\,du$

$= 2(2u^{\frac{1}{2}}) + c$

$= 4\sqrt{u} + c$

$= 4\sqrt{x^2 + 3x} + c$

g $\int \frac{2x-5}{(x^2-5x+2)^6}\,dx$

Let $u = x^2 - 5x + 2$

$\frac{du}{dx} = 2x - 5$ or $dx = \frac{du}{2x-5}$

So $\int \frac{2x-5}{(x^2-5x+2)^6}\,dx$

$= \int \frac{2x-5}{(u)^6} \times \frac{du}{2x-5}$

$= \int (u)^{-6}\,du$

$= \frac{1}{-5}u^{-5} + c$

$= \frac{-(x^2-5x+2)^{-5}}{5} + c$

h $\int (x^2-1)\sqrt{4-3x+x^3}\,dx$

Let $u = 4 - 3x + x^3$

$\frac{du}{dx} = -3 + 3x^2 = 3x^2 - 3$ or $dx = \frac{du}{3x^2-3}$

So $\int (x^2-1)\sqrt{4-3x+x^3}\,dx$

$= \int (x^2-1)\sqrt{u}\,\frac{du}{3x^2-3}$

$= \int (x^2-1)\sqrt{u}\,\frac{du}{3(x^2-1)}$

$= \frac{1}{3}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{3}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c$

$= \frac{2}{9}u^{\frac{3}{2}} + c$

$= \frac{2(4-3x+x^3)^{\frac{3}{2}}}{9} + c$

i $\int (6x-3)e^{x^2-x+3}\,dx$

Let $u = x^2 - x + 3$

$\frac{du}{dx} = 2x - 1$ or $dx = \frac{du}{2x-1}$

So $\int (6x-3)e^{x^2-x+3}\,dx$

$= \int (6x-3)e^u\,\frac{du}{2x-1}$

$= 3\int (2x-1)e^u\,\frac{du}{2x-1}$

$= 3\int e^u\,du$

$= 3e^u + c$

$= 3e^{x^2-x+3} + c$

j $\int x^2 e^{x^3+2}\,dx$

Let $u = x^3 + 2$

$\frac{du}{dx} = 3x^2$ or $dx = \frac{du}{3x^2}$

So $\int x^2 e^{x^3+2}\,dx$

$= \int x^2 e^u\,\frac{du}{3x^2}$

$= \frac{1}{3}\int e^u\,du$

$= \frac{e^u}{3} + c$

$= \frac{e^{x^3+2}}{3} + c$

k $\int (x+1)\sin(x^2+2x-3)\,dx$

Let $u = x^2 + 2x$

$\frac{du}{dx} = 2x + 2$ or $dx = \frac{du}{2x+2}$

So $\int (x+1)\sin(x^2+2x-3)\,dx$

$= \int (x+1)\sin(u)\frac{du}{2x+2}$

$= \int (x+1)\sin(u)\frac{du}{2(x+1)}$

$= \frac{1}{2}\int \sin(u)\,du$

$= \frac{1}{2}(-\cos(u)) + c$

$= -\frac{\cos(u)}{2} + c$

$= \frac{-\cos(x^2+2x-3)}{2} + c$

l $\int (x^2-2)\cos(6x-x^3)\,dx$

Let $u = 6x - x^3$

$\frac{du}{dx} = 6 - 3x^2$ or $dx = \frac{du}{6-3x^2}$

So $\int (x^2-2)\cos(6x-x^3)\,dx$

$= \int (x^2-2)\cos(u)\frac{du}{6-3x^2}$

$= \int (x^2-2)\cos(u)\frac{du}{-3(x^2-2)}$

$= \frac{-1}{3}\int \cos(u)\,du$

$= \frac{-1}{3}\sin(u) + c$

$= \frac{-\sin(6x-x^3)}{3} + c$

m $\int \sin(2x)\cos^4(2x)\,dx$

Let $u = \cos(2x)$

$\frac{du}{dx} = -2\sin(2x)$ or $dx = \frac{du}{-2\sin(2x)}$

$\int \sin(2x)\cos^4(2x)\,dx$

$= \int \sin(2x)(u)^4\,\frac{du}{-2\sin(2x)}$

$= \frac{-1}{2}\int u^4\,du$

$= \frac{-1}{2}\left(\frac{1}{5}u^5\right) + c$

$= \frac{-u^5}{10} + c$

$= \frac{-\cos^5(2x)}{10} + c$

n $\int \cos(3x)\sin^2(3x)\,dx$

Let $u = \sin(3x)$

$\frac{du}{dx} = 3\cos(3x)$ or $dx = \frac{du}{3\cos(3x)}$

So $\int \cos(3x)\sin^2(3x)\,dx$

$= \int \cos(3x)(u)^2 \frac{du}{3\cos(3x)}$

$= \frac{1}{3}\int u^2\,du$

$= \frac{1}{3}\left(\frac{1}{3}u^3\right) + c$

$= \frac{u^3}{9} + c$

$= \frac{\sin^3(3x)}{9} + c$

o $\int \frac{\log_e(3x)}{2x}\,dx$

Let $u = \log_e(3x)$

$\frac{du}{dx} = \frac{1}{x}$ or $dx = x\,du$

So $\int \frac{\log_e(3x)}{2x}\,dx$

$= \int \frac{u}{2x} x\,du$

$= \frac{1}{2}\int u\,du$

$= \frac{1}{2}\left(\frac{1}{2}u^2\right) + c$

$= \frac{u^2}{4} + c$

$= \frac{(\log_e(3x))^2}{4} + c$

p $\int \frac{(4x-2)\log_e(x^2-x)}{x^2-x}\,dx$

Let $u = x^2 - x$

$\frac{du}{dx} = 2x - 1$ or $dx = \frac{du}{2x-1}$

So $\int \frac{(4x-2)\log_e(x^2-x)}{x^2-x}\,dx$

$= \int \frac{2(2x-1)\log_e(u)}{u}\,\frac{du}{2x-1}$

$= 2\int \frac{\log_e(u)}{u}\,du$

Let $v = \log_e(u)$

$\frac{dv}{du} = \frac{1}{u}$ or $du = u\,dv$

So $\int \frac{(4x-2)\log_e(x^2-x)}{x^2-x}\,dx$

$= 2\int \frac{\log_e(u)}{u}\,du$

$= 2\int \frac{v}{u} u\,dv$

$= 2\int v\,dv$

$= 2\left(\frac{1}{2}v^2\right) + c$

$= v^2 + c$

$= (\log_e(u))^2 + c$

$= [\log_e(x^2 - x)]^2 + c$

5 a $\int x(x^2+1)^{\frac{5}{2}}\,dx$

Let $u = x^2 + 1$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

So $\int x(x^2+1)^{\frac{5}{2}}\,dx$

$= \int x(u)^{\frac{5}{2}}\frac{du}{2x}$

$= \frac{1}{2}\int u^{\frac{5}{2}}du$

$= \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}}\right) + c$

$= \frac{u^{\frac{7}{2}}}{7} + c$

$= \frac{(x^2+1)^{\frac{7}{2}}}{7} + c$

b $\int x\sqrt{1-x^2}\,dx$

Let $u = 1 - x^2$

$\frac{du}{dx} = -2x$ or $dx = \frac{du}{-2x}$

So $\int x\sqrt{u}\,\frac{du}{-2x}$

$= \frac{-1}{2}\int u^{\frac{1}{2}}du$

$= \frac{-1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c$

$= \frac{-u^{\frac{3}{2}}}{3} + c$

$= \frac{-(1-x^2)^{\frac{3}{2}}}{3} + c$

c $\int e^x(3+2e^x)^4\,dx$

Let $u = 3 + 2e^x$

$\frac{du}{dx} = 2e^x$ or $dx = \frac{du}{2e^x}$

So $\int e^x(3+2e^x)^4\,dx$

$= \int e^x(u)^4\frac{du}{2e^x}$

$= \frac{1}{2}\int u^4 du$

$= \frac{1}{2}\left(\frac{1}{5}u^5\right) + c$

$= \frac{u^5}{10} + c$

$= \frac{(3+2e^x)^5}{10} + c$

d $\int \frac{\sin(x)}{\cos^3(x)}\,dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

So $\int \frac{\sin(x)}{\cos^3(x)} dx$

$= \int \frac{\sin(x)}{(u)^3} \times \frac{du}{-\sin(x)}$

$= -\int u^{-3} du$

$= -\left(\frac{1}{-2}u^{-2}\right) + c$

$= \frac{1}{2u^2} + c$

$= \frac{1}{2\cos^2(x)} + c$

e $\int x^2 \sin(x)^3 dx$

Let $u = x^3$

$\frac{du}{dx} = 3x^2$ or $dx = \frac{du}{3x^2}$

So $\int x^2 \sin(x^3)\, dx$

$= \int x^2 \sin(u) \frac{du}{3x^2}$

$= \frac{1}{3}\int \sin(u) du$

$= -\frac{\cos(u)}{3} + c$

$= \frac{-\cos(x)^3}{3} + c$

f $\int \sin(x) e^{\cos(x)} dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

So $\int \sin(x) e^{\cos(x)} dx$

$= \int \sin(x) e^u \frac{du}{-\sin(x)}$

$= -\int u^u du$

$= -e^u + c$

$= -e^{\cos(x)} + c$

g $\int \frac{\cos(x)\log_e(\sin(x))}{\sin(x)} dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

So $\int \frac{\cos(x)\log_e(\sin(x))}{\sin(x)} dx$

$= \int \frac{\cos(x)\log_e(u)}{u} \times \frac{du}{\cos(x)}$

$= \int \frac{\log_e(u)}{u} du$

Let $v = \log_e(u)$

$\frac{dv}{du} = \frac{1}{u}$ or $du = x\, dv$

So $\int \frac{\cos(x)\log_e(\sin(x))}{\sin(x)} dx$

$= \int \frac{\log_e(u)}{u} du$

$= \int \frac{v}{u} u\, dv$

$= \int v\, dv$

$= \frac{1}{2}v^2 + c$

$= \frac{(\log_e(u))^2}{2} + c$

$= \frac{[\log_e(\sin(x))]^2}{2} + c$

h $\int e^{3x}(1-e^{3x})^2 dx$

Let $u = 1 - e^{3x}$

$\frac{du}{dx} = -3e^{3x}$ or $dx = \frac{du}{-3e^{3x}}$

So $\int e^{3x}(1-e^{3x})^2 dx$

$= \int e^{3x}(u)^2 \frac{du}{-3e^{3x}}$

$= \frac{-1}{3}\int u^2\, du$

$= \frac{-1}{3}\left(\frac{1}{3}u^3\right) + c$

$= \frac{-u^3}{9} + c$

$= \frac{-(1-e^{3x})^3}{9} + c$

i $\int \frac{-2\cos^{-1}\left(\frac{x}{3}\right)}{\sqrt{9-x^2}} dx$

Let $u = \cos^{-1}\left(\frac{x}{3}\right)$

$\frac{du}{dx} = \frac{-1}{\sqrt{3^2 - x^2}}$

$\frac{du}{dx} = \frac{-1}{\sqrt{9-x^2}}$ or $dx = -\sqrt{9-x^2}\, du$

So $\int \frac{-2\cos^{-1}\left(\frac{x}{3}\right)}{\sqrt{9-x^2}} dx$

$= \int \frac{-2u}{\sqrt{9-x^2}} \times \left(-\sqrt{9-x^2}\right) du$

$= 2\int u\, du$

$= 2\left(\frac{1}{2}u^2\right) + c$

$= u^2 + c$

$= \left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + c$

j $\int (2x+1)\sqrt{x+x^2-3}\, dx$

Let $u = x + x^2 - 3$

$\frac{du}{dx} = 1 + 2x = 2x + 1$ or $dx = \frac{du}{2x+1}$

So $\int (2x+1)\sqrt{x+x^2-3}\, dx$

$= \int (2x+1)\sqrt{u}\, \frac{du}{2x+1}$

$= \int u^{\frac{1}{2}}\, du$

$= \frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2(x + x^2 + 3)^{\frac{3}{2}}}{3} + c$

k $\int (x+2)\cos(x^2+4x)\ dx$

Let $u = x^2 + 4x$

$\frac{du}{dx} = 2x + 4$ or $dx = \frac{du}{2x+4}$

So $\int (x+2)\cos(x^2+4x)\ dx$

$= \int (x+2)\cos(u)\ \frac{du}{2x+4}$

$= \int (x+2)\cos(u)\ \frac{du}{2(x+2)}$

$= \frac{1}{2}\int \cos(u)\ du$

$= \frac{\sin(u)}{2} + c$

$= \frac{\sin(x^2+4x)}{2} + c$

l $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\ dx$

Let $u = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}(x+1)^{\frac{-1}{2}}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x+1}}$ or $dx = 2\sqrt{x+1}\ du$

So $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\ dx$

$= \int \frac{e^u}{\sqrt{x+1}} \times 2\sqrt{x+1}\ du$

$= 2\int e^u\ du$

$= 2e^u + c$

$= 2e^{\sqrt{x+1}} + c$

m $\int \frac{\sin^{-1}(4x)}{\sqrt{1-16x^2}}\ dx$

Let $u = \sin^{-1}(4x)$

$\frac{du}{dx} = \frac{1}{\sqrt{\left(\frac{1}{4}\right)^2 - x^2}}$

$\frac{du}{dx} = \frac{1}{\sqrt{\frac{1-16x^2}{16}}}$

$\frac{du}{dx} = \frac{1}{\frac{1}{4}\sqrt{1-16x^2}}$

$\frac{du}{dx} = \frac{4}{\sqrt{1-16x^2}}$ or $dx = \sqrt{\frac{1-16x^2}{4}}\ du$

So $\int \frac{\sin^{-1}(4x)}{\sqrt{1-16x^2}}\ dx$

$= \frac{u}{\sqrt{1-16x^2}} \times \sqrt{\frac{1-16x^2}{4}}\ du$

$= \frac{1}{4}\int u\ du$

$= \frac{1}{4}\left(\frac{1}{2}u^2\right) + c$

$= \frac{u^2}{8} + c$

$= \frac{(\sin^{-1}(4x))^2}{8} + c$

n $\int \frac{\tan^{-1}(x)}{1+x^2}\ dx$

Let $u = \tan^{-1}(x)$

$\frac{du}{dx} = \frac{1}{1+x^2}$ or $dx = (1 + x^2)\ du$

So $\int \frac{\tan^{-1}(x)}{1+x^2}\ dx$

$= \int \frac{u}{1+x^2}(1+x^2)\ du$

$= \int u\ du$

$= \frac{1}{2}u^2 + c$

$= \frac{(\tan^{-1} x)^2}{2} + c$

o $\int \frac{x}{1-4x^2}\ dx$

Let $u = 1 - 4x^2$

$\frac{du}{dx} = -8x$ or $dx = \frac{du}{-8x}$

So $\int \frac{x}{1-4x^2}\ dx$

$= \int \frac{x}{u} \times \frac{du}{-8x}$

$= \frac{-1}{8}\int \frac{1}{u}\ du$

$= \frac{-1}{8}\log_e(u) + c$

$= \frac{-\log_e(1-4x^2)}{8} + c$

6 a $\int \frac{\cos(x)}{\sqrt{1+3\sin(x)}}\ dx$

Let $u = 1 + 3\sin(x)$

$\frac{du}{dx} = 3\cos(x)$ or $dx = \frac{du}{3\cos(x)}$

So $\int \frac{\cos(x)}{\sqrt{1+3\sin(x)}}\ dx$

$= \int \frac{\cos(x)}{\sqrt{u}}\ \frac{du}{3\cos(x)}$

$= \frac{1}{3}\int u^{\frac{-1}{2}}\ du$

$= \frac{1}{3}(2u^{\frac{1}{2}}) + c$

$= \frac{2}{3}u^{\frac{1}{2}} + c$

$= \frac{2(1+3\sin(x))^{\frac{1}{2}}}{3} + c$

$= \frac{2\sqrt{1+3\sin(x)}}{3} + c$

b $\int \sec^2(x)\sqrt{2+\tan(x)}\ dx$

Let $u = 2 + \tan(x)$

$\frac{du}{dx} = \sec^2(x)$ or $dx = \frac{du}{\sec^2(x)}$

So $\int \sec^2(x)\sqrt{2+\tan(x)}\ dx$

$= \int \sec^2(x)\sqrt{u}\ \frac{du}{\sec^2(x)}$

$= \int u^{\frac{1}{2}}\ du$

$= \frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2(2+\tan(x))^{\frac{3}{2}}}{3} + c$

c $\int \sin(x)\sec^3(x)\ dx$

Let $u = \sec(x)$

$u = (\cos(x))^{-1}$

$\frac{du}{dx} = -(\cos(x))^{-2}(-\sin(x))$

$= \frac{\sin(x)}{\cos^2(x)}$

$= \sin(x)\sec^2(x)$ or $dx = \frac{du}{\sin(x)\sec^2(x)}$

So $\int \sin(x)\sec^3(x)\ dx$

$= \int \sin(x)u^3\ \frac{du}{\sin(x)\sec^2(x)}$

$= \int \frac{u^3}{u^2}\ du$

$= \int u\ du$

$= \frac{1}{2}u^2 + c$

$= \frac{1}{2}(\sec(x))^2 + c$

$= \frac{\sec^2(x)}{2} + c$

d $\int \frac{e^{2x}}{(e^{2x}-3)^2}\ dx$

Let $u = e^{2x} - 3$

$\frac{du}{dx} = 2e^{2x}$ or $dx = \frac{du}{2e^{2x}}$

So $\int \frac{e^{2x}}{(e^{2x}-3)^2}\ dx$

$= \int \frac{e^{2x}}{(u)^2} \times \frac{du}{2e^{2x}}$

$= \frac{1}{2}\int u^{-2}\ du$

$= \frac{1}{2}\left(\frac{1}{-1}u^{-1}\right) + c$

$= \frac{-1}{2u} + c$

$= \frac{-1}{2(e^{2x}-3)} + c$

e $\int \frac{\sec^2(x)}{(5-\tan(x))^3}\ dx$

Let $u = 5 - \tan(x)$

$\frac{du}{dx} = -\sec^2(x)$ or $dx = \frac{du}{-\sec^2(x)}$

So $\int \frac{\sec^2(x)}{(5-\tan(x))^3}\ dx$

$= \int \frac{\sec^2(x)}{(u)^3} \times \frac{du}{-\sec^2(x)}$

$= -\int u^{-3}\ du$

$= -\left(\frac{1}{-2}u^{-2}\right) + c$

$= \frac{1}{2u} + c$

$= \frac{1}{2(5-\tan(x))^2} + c$

f $\int \frac{4}{x\log_e(x)}\ dx$

Let $u = \log_e(x)$

$\frac{du}{dx} = \frac{1}{x}$ or $dx = x\ du$

So $\int \frac{4}{x\log_e(x)}\ dx$

$= \int \frac{4}{xu}x\ du$

$= 4\int \frac{1}{u}\ du$

$= 4\log_e(u) + c$

$= 4\log_e(\log_e(x)) + c$

g $\int \frac{(\log_e(x))^3}{x}\ dx$

Let $u = \log_e(x)$

$\frac{du}{dx} = \frac{1}{x}$ or $dx = x\ du$

So $\int \frac{(\log_e(x))^3}{x}\ dx$

$= \int \frac{(u)^3}{x}x\ du$

$= \int u^3\ du$

$= \frac{1}{4}u^4 + c$

$= \frac{(\log_e(x))^4}{4} + c$

h $\int \frac{e^{\tan(x)}}{\cos^2(x)}\ dx$

Let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x) = \frac{1}{\cos^2(x)}$ or $dx = \cos^2(x)\ du$

So $\int \frac{e^{\tan(x)}}{\cos^2(x)}\ dx$

$= \int \frac{e^4}{\cos^2(x)}\cos^2(x)\ du$

$= \int e^4\ du$

$= e^4 + c$

$= e^{\tan(x)} + c$

i $\int \frac{e^x - e^{-x}}{\sqrt{e^x + e^{-x}}}\ dx$

Let $u = e^x + e^{-x}$

$\frac{du}{dx} = e^x - e^{-x}$ or $dx = \frac{du}{e^x - e^{-x}}$

So $\int \frac{e^x - e^{-x}}{\sqrt{e^x + e^{-x}}}\, dx$

$= \int \frac{e^x - e^{-x}}{\sqrt{u}} \times \frac{du}{e^x - e^{-x}}$

$= \int u^{\frac{-1}{2}}\, du$

$= 2u^{\frac{1}{2}} + c$

$= 2(e^x - e^{-x})^{\frac{1}{2}} + c$

$= 2\sqrt{e^x - e^{-x}} + c$

j $\int \frac{\sin(x) - \cos(x)}{\sin(x) + \cos(x)}\, dx$

Let $u = \sin(x) + \cos(x)$

$\frac{du}{dx} = \cos(x) - \sin(x)$

$\frac{du}{dx} = -(\sin(x) - \cos(x))$ or $dx = \frac{du}{-(\sin(x) - \cos(x))}$

So $\int \frac{\sin(x) - \cos(x)}{\sin(x) + \cos(x)}\, dx$

$= \int \frac{\sin(x) - \cos(x)}{u} \times \frac{du}{-(\sin(x) - \cos(x))}$

$= -\int \frac{1}{u}\, du$

$= -\log_e(u) + c$

$= -\log_e((\sin(x) + \cos(x)) + c$

k $\int \sin^3(x)\cos^2(x)\, dx$

$= \int \sin(x)(1 - \cos^2(x))\cos^2(x)\, dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

So $\int \sin^3(x)\cos^2(x)\, dx$

$= \int \sin(x)(1 - u^2)u^2 \frac{du}{-\sin(x)}$

$= \int u^2(u^2 - 1)\, du$

$= \int u^4 - u^2\, du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + c$

$= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + c$

l $\int \cos^3(x)\sin^4(x)\, dx$

$= \int \cos(x)(1 - \sin^2(x))\sin^4(x)\, dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

So $\int \cos^3(x)\sin^4(x)\, dx$

$= \int \cos(x)(1 - u^2)u^4 \frac{du}{\cos(x)}$

$= \int u^4(1 - u^2)\, du$

$= \int u^4 - u^6\, du$

$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + c$

$= \frac{1}{5}\sin^5(x) - \frac{1}{7}\sin^7(x) + c$

m $\int \frac{\log_e(\tan(x))}{\sin(x)\cos(x)}\, dx$

Let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$

$\frac{du}{dx} = \frac{1}{\cos^2(x)}$ or $dx = \cos^2(x)\, du$

So $\int \frac{\log_e(\tan(x))}{\sin(x)\cos(x)}\, dx$

$= \int \frac{\log_e(u)}{\sin(x)\cos(x)} \cos^2(x)\, du$

$= \int \frac{\log_e(u)\cos(x)}{\sin(x)}\, du$

$= \int \frac{\log_e(u)}{\tan(x)}\, du$

$= \int \frac{\log_e(u)}{u}\, du$

Let $v = \log_e(u)$

$\frac{dv}{du} = \frac{1}{u}$ or $du = u\, dv$

So $\int \frac{\log_e(\tan(x))}{\sin(x)\cos(x)}\, dx$

$= \int \frac{\log_e(u)}{u}\, du$

$= \int \frac{v}{u}\, u\, dv$

$= \int v\, dv$

$= \frac{1}{2}v^2 + c$

$= \frac{(\log_e(u))^2}{2} + c$

$= \frac{[\log_e(\tan(x))]^2}{2} + c$

7 If $f'(x) = \frac{x}{\sqrt{x^2+5}}$ and $f(2) = 1$, find $f(x)$

$f'(x) = \frac{x}{\sqrt{x^2+5}}$

$f(x) = \int \frac{x}{\sqrt{x^2+5}}\, dx$

Let $u = x^2 + 5$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

So $f(x) = \int \frac{x}{\sqrt{u}} \times \frac{du}{2x}$

$= \frac{1}{2}\int u^{\frac{-1}{2}}\, du$

$= \frac{1}{2}(2u^{\frac{1}{2}}) + c$

$= u^{\frac{1}{2}} + c$

$= (x^2 + 5)^{\frac{1}{2}} + c$

$f(x) = \sqrt{x^2+5}+c$
$f(2) = \sqrt{2^2+5}+c = 1$
$\sqrt{9}+c = 1$
$3 + c = 1$
$c = -2$

Therefore $f(x) = \sqrt{x^2+5}-2$

8 If $f'(x) = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$ and $f(0) = 3$, find $f(x)$

$f'(x) = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$

$f(x) = \int \dfrac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$

Let $u = \sqrt{x} = x^{\frac{1}{2}}$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{2}x^{\frac{-1}{2}}$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{2\sqrt{x}}$ or $\mathrm{d}x = 2\sqrt{x}\,\mathrm{d}u$

So $f(x) = \int \dfrac{e^u}{\sqrt{x}} \times 2\sqrt{x}\,\mathrm{d}u$
$= 2\int e^u\,\mathrm{d}u$
$= 2e^u + c$
$f(x) = 2e^{\sqrt{x}}+c$
$f(0) = 2e^0 + c = 3$
$2 + c = 3$
$c = 1$

Therefore $f(x) = 2e^{\sqrt{x}}+1$

9 If $g(1) = -2$ and $g'(x) = \dfrac{4\log_e(x^2)}{x}$ then find $g(x)$

$g'(x) = \dfrac{4\log_e(x^2)}{x}$

$g(x) = \int \dfrac{4\log_e(x^2)}{x}\,\mathrm{d}x$

Let $u = x^2$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$ or $\mathrm{d}x = \dfrac{\mathrm{d}u}{2x}$

So $g(x) = \int \dfrac{4\log_e(u)}{x}\times\dfrac{\mathrm{d}u}{2x}$
$= \int \dfrac{2\log_e(u)}{x^2}\,\mathrm{d}u$
$= \int \dfrac{2\log_e(u)}{u}\,\mathrm{d}u$

Let $v = \log_e(u)$

$\dfrac{\mathrm{d}v}{\mathrm{d}u} = \dfrac{1}{u}$ or $\mathrm{d}u = u\,\mathrm{d}v$

So $g(x) = \int \dfrac{2v}{u}u\,\mathrm{d}v$
$= 2\int v\,\mathrm{d}v$
$= 2\left(\dfrac{1}{2}v^2\right)+c$
$= v^2 + c$
$= (\log_e(u))^2 + c$
$g(x) = (\log_e(x^2))^2 + c$
$g(1) = (\log_e(1^2))^2 + c = -2$
$0^2 + c = -2$
$c = -2$

So $g(x) = (\log_e(x^2))^2 - 2$

10 If $g\left(\dfrac{\pi}{4}\right) = 0$ and $g'(x) = 16\sin(x)\cos^3(x)$ then find $g(x)$

$g'(x) = 16\sin(x)\cos^3(x)$
$g(x) = \int 16\sin(x)\cos^3(x)$

Let $u = \cos(x)$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = -\sin(x)$ or $\mathrm{d}x = \dfrac{\mathrm{d}u}{-\sin(x)}$

So $g(x) = \int 16\sin(x)u^3\dfrac{\mathrm{d}u}{-\sin(x)}$
$= -16\int u^3\,\mathrm{d}u$
$= -16\left(\dfrac{1}{4}u^4\right)+c$
$= -4u^4 + c$
$g(x) = -4\cos^4(x) + c$
$g\left(\dfrac{\pi}{4}\right) = -4\left(\cos\left(\dfrac{\pi}{4}\right)\right)^4 + c = 0$
$-4\left(\dfrac{1}{\sqrt{2}}\right)^4 + c = 0$
$-4\times\dfrac{1}{4} + c = 0$
$-1 + c = 0$
$c = 1$

Therefore $g(x) = 1 - 4\cos^4(x)$

Exercise 4C — Linear substitution

1 a i $\int \dfrac{4}{x-3}\,\mathrm{d}u$

Let $u = x - 3$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$ or $\mathrm{d}x = \mathrm{d}u$

So $\int \dfrac{4}{x-3}\,\mathrm{d}u$
$= \int \dfrac{4}{u}\,\mathrm{d}u$

ii $= 4\log_e(u) + c$
$= 4\log_e(x-3) + c$

b i $\int \dfrac{2}{3x+5}\,\mathrm{d}x$

Let $u = 3x + 5$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 3$ or $\mathrm{d}x = \dfrac{\mathrm{d}u}{3}$

So $\int \dfrac{2}{3x+5}\,\mathrm{d}x$
$= \int \dfrac{2}{u}\times\dfrac{\mathrm{d}u}{3}$
$= \int \dfrac{2}{3u}\,\mathrm{d}u$

ii $= \dfrac{2}{3}\log_e(u) + c$
$= \dfrac{2\log_e(3x+5)}{3} + c$

c i $\int \sqrt{4x+1}\,\mathrm{d}x$

Let $u = 4x + 1$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 4$ or $\mathrm{d}x = \dfrac{\mathrm{d}u}{4}$

So $\int \sqrt{4x+1}\,\mathrm{d}x$

$= \int \sqrt{u}\ \frac{du}{4}$

$= \int \frac{\sqrt{u}}{4} du$

ii $= \frac{1}{4}\int u^{\frac{1}{2}}\ du$

$= \frac{1}{4}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c$

$= \frac{u^{\frac{3}{2}}}{6} + c$

$= \frac{(4x+1)^{\frac{3}{2}}}{6} + c$

d i $\int \sqrt{3-2x}\ dx$

Let $u = 3 - 2x$

$\frac{du}{dx} = -2$ or $dx = \frac{du}{-2}$

So $\int \sqrt{3-2x}\ dx$

$= \int \sqrt{u}\ \frac{du}{-2}$

$= \int \frac{-\sqrt{u}}{2}\ du$

ii $= \frac{-1}{2}\int u^{\frac{1}{2}}\ du$

$= \frac{-1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c$

$= \frac{-u^{\frac{3}{2}}}{3} + c$

$= \frac{-(3-2x)^{\frac{3}{2}}}{3} + c$

e i $\int x(x+1)^3\ dx$

Let $u = x + 1$ and $x = u - 1$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int x(x+1)^3\ dx$

$= \int (u-1)(u)^3\ du$

$= \int (u^4 - u^3)\ du$

ii $= \frac{1}{5}u^5 - \frac{1}{4}u^4 + c$

$= \frac{(x+1)^5}{5} - \frac{(x+1)^4}{4} + c$

$= \frac{4(x+1)^5 - 5(x+1)^4}{20} + c$

$= \frac{[4(x+1) - 5]\ (x+1)^4}{20} + c$

$= \frac{(4x+4-5)(x+1)^4}{20} + c$

$= \frac{(4x-1)(x+1)^4}{20} + c$

f i $\int 4x(x-3)^4\ dx$

Let $u = x - 3$ and $x = u + 3$

$\frac{du}{dx} = 1$ or $du = dx$

So $\int 4x(x-3)^4\ dx$

$= \int 4(u+3)(u)^4\ du$

$= \int 4u^4(u+3)\ du$

$= \int (4u^5 + 12u^4)\ du$

ii $= \frac{4}{6}u^6 + \frac{12}{5}u^5 + c$

$= \frac{2}{3}(x-3)^6 + \frac{12}{5}(x-3)^5 + c$

$= \frac{10(x-3)^6 + 36(x-3)^5}{15} + c$

$= \frac{2\,[5(x-3) + 18]\,(x-3)^5}{15} + c$

$= \frac{2\,(5x-15+18)(x-3)^5}{15} + c$

$= \frac{2(5x+3)(x-3)^5}{15} + c$

g i $\int 2x(2x+1)^4\ dx$

Let $u = 2x + 1$ and $2x = u - 1$

$\frac{du}{dx} = 2$ or $dx = \frac{du}{2}$

So $\int 2x(2x+1)^4 dx$

$= \int (u-1)(u)^4\ \frac{du}{2}$

$= \int \frac{u^5 - u^4}{2}\ du$

ii $= \frac{1}{2}\int (u^5 - u^4)\ du$

$= \frac{1}{2}\left(\frac{1}{6}u^6 - \frac{1}{5}u^5\right) + c$

$= \frac{1}{2}\left[\frac{(2x+1)^6}{6} - \frac{(2x+1)^5}{5}\right] + c$

$= \frac{1}{2}\left[\frac{5(2x+1)^6 - 6(2x+1)^5}{30}\right] + c$

$= \frac{[5(2x+1) - 6]\ (2x+1)^5}{60} + c$

$= \frac{(10x+5-6)\ (2x+1)^5}{60} + c$

$= \frac{(10x-1)\ (2x+1)^5}{60} + c$

h i $\int 3x(1-3x)^5\ dx$

Let $u = 1 - 3x$ so $3x = 1 - u$

$\frac{du}{dx} = -3$ or $dx = \frac{du}{-3}$

So $\int 3x(1-3x)^5\ dx$

$= \int (1-u)(u)^5\ \frac{du}{-3}$

$= \int \frac{u^5(u-1)}{3}\ du$

$= \int \frac{u^6 - u^5}{3}\ du$

ii $= \frac{1}{3}\int (u^6 - u^5)\ du$

$= \frac{1}{3}\left(\frac{1}{7}u^7 - \frac{1}{6}u^6\right) + c$

$= \frac{1}{3}\left(\frac{6u^7 - 7u^6}{42}\right) + c$

$= \frac{(6u-7)u^6}{126} + c$

$= \frac{[6(1-3x) - 7]\,(1-3x)^6}{126} + c$

$= \frac{(6-18x-7)(1-3x)^6}{126} + c$

$= \frac{-(18x+1)(1-3x)^6}{126} + c$

i i $\int 6x(3x-2)^{\frac{3}{4}}\ dx$

Let $u = 3x - 2$ and $3x = u + 2$

$\frac{du}{dx} = 3$ or $dx = \frac{du}{3}$

So $\int 6x(3x-2)^{\frac{3}{4}}\ dx$

$= \int 2(3x)(u)^{\frac{3}{4}}\ \frac{du}{3}$

$= \int \frac{2}{3}(u+2)\ u^{\frac{3}{4}}\ du$

$= \int \frac{2}{3}(u^{\frac{7}{4}} + 2u^{\frac{3}{4}})\ du$

ii $= \frac{2}{3}\int u^{\frac{7}{4}} + 2u^{\frac{3}{4}}\ du$

$= \frac{2}{3}\left(\frac{4}{11}u^{\frac{11}{4}} + \frac{8}{7}u^{\frac{7}{4}}\right) + c$

$= \frac{8}{3}\left(\frac{u^{\frac{11}{4}}}{11} + \frac{2u^{\frac{7}{4}}}{7}\right) + c$

$= \frac{8}{3}\left(\frac{7u^{\frac{11}{4}} + 22u^{\frac{7}{4}}}{77}\right) + c$

$= \frac{8}{231}u^{\frac{7}{4}}(7u + 22) + c$

$= \frac{8}{231}(3x-2)^{\frac{7}{4}}[7(3x-2) + 22] + c$

$= \frac{8}{231}(3x-2)^{\frac{7}{4}}(21x - 14 + 22) + c$

$= \frac{8}{231}(3x-2)^{\frac{7}{4}}(21x + 8) + c$

j i $\int x(2x+7)^{\frac{1}{3}}\ dx$

Let $u = 2x + 7$ and $x = \frac{u-7}{2}$

$\frac{du}{dx} = 2$ or $dx = \frac{du}{2}$

So $\int x(2x+7)^{\frac{1}{3}}\ dx$

$= \int \frac{(u-7)}{2}\,(u)^{\frac{1}{3}}\ \frac{du}{2}$

$= \int \frac{u^{\frac{1}{3}}(u-7)}{4}\ du$

$= \int \frac{u^{\frac{4}{3}} - 7u^{\frac{1}{3}}}{4}\ du$

ii $= \frac{1}{4}\int \left(u^{\frac{4}{3}} - 7u^{\frac{1}{3}}\right) du$

$= \frac{1}{4}\left(\frac{3}{7}u^{\frac{7}{3}} - \frac{7\times 3}{4}u^{\frac{4}{3}}\right) + c$

$= \frac{3}{4}\left(\frac{u^{\frac{7}{3}}}{7} - \frac{7u^{\frac{4}{3}}}{4}\right) + c$

$= \frac{3}{4}\left(\frac{4u^{\frac{7}{3}} - 49u^{\frac{4}{3}}}{28}\right) + c$

$= \frac{3(4u-49)u^{\frac{4}{3}}}{112} + c$

$= \frac{3\,[4(2x+7) - 49](2x+7)^{\frac{4}{3}}}{112} + c$

$= \frac{3\,(8x + 28 - 49)(2x+7)^{\frac{4}{3}}}{112} + c$

$= \frac{3\,(8x - 21)(2x+7)^{\frac{4}{3}}}{112} + c$

k i $\int x\sqrt{x+3}\ dx$

Let $u = x + 3$ and $x = u - 3$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int x\sqrt{x+3}\,dx$

$= \int (u-3)\sqrt{u}\ du$

$= \int u^{\frac{1}{2}}(u-3)\,du$

$= \int (u^{\frac{3}{2}} - 3u^{\frac{1}{2}})\,du$

ii $= \frac{2}{5}u^{\frac{5}{2}} - \frac{2\times 3}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{5}(u^{\frac{5}{2}} - 5u^{\frac{3}{2}}) + c$

$= \frac{2}{5}(u-5)u^{\frac{3}{2}} + c$

$= \frac{2}{5}(x+3-5)(x+3)^{\frac{3}{2}} + c$

$= \frac{2(x-2)(x+3)^{\frac{3}{2}}}{5} + c$

l i $\int x\sqrt{3x-4}\,dx$

Let $u = 3x - 4$ and $x = \frac{u+4}{3}$

$\frac{du}{dx} = 3$ or $dx = \frac{du}{3}$

So $\int x\sqrt{3x-4}\,dx$

$= \int \frac{u+4}{3}\sqrt{u}\,\frac{du}{3}$

$= \int \frac{u^{\frac{1}{2}}(u+4)}{9}\,du$

$= \int \frac{u^{\frac{3}{2}}+4u^{\frac{1}{2}}}{9}du$

ii $= \frac{1}{9}\int (u^{\frac{5}{2}}+4u^{\frac{1}{2}})du$

$= \frac{1}{9}\left(\frac{2}{5}u^{\frac{5}{2}}+\frac{4\times 2}{3}u^{\frac{3}{2}}\right)+c$

$= \frac{2}{9}\left(\frac{u^{\frac{5}{2}}}{5}+\frac{4u^{\frac{3}{2}}}{3}\right)+c$

$= \frac{2}{9}\left(\frac{3u^{\frac{5}{2}}+20u^{\frac{3}{2}}}{15}\right)+c$

$= \frac{2}{135}u^{\frac{3}{2}}(3u+20)+c$

$= \frac{2}{135}(3x-4)^{\frac{3}{2}}[3(3x-4)+20]+c$

$= \frac{2}{135}(3x-4)^{\frac{3}{2}}(9x-12+20)+c$

$= \frac{2}{135}(3x-4)^{\frac{3}{2}}(9x+8)+c$

m i $\int (x+2)(x-4)^{\frac{3}{2}}\,dx$

Let $u = x-4$ and $x = u+4$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int (x+2)(x-4)^{\frac{3}{2}}dx$

$= \int (u+4+2)(u)^{\frac{3}{2}}\,du$

$= \int (u+6)u^{\frac{3}{2}}\,du$

$= \int (u^{\frac{5}{2}}+6u^{\frac{3}{2}})du$

ii $= \frac{2}{7}u^{\frac{7}{2}}+\frac{6\times 2}{5}u^{\frac{5}{2}}+c$

$= 2\left(\frac{u^{\frac{7}{2}}}{7}+\frac{6u^{\frac{5}{2}}}{5}\right)+c$

$= 2\left(\frac{5u^{\frac{7}{2}}+42u^{\frac{5}{2}}}{35}\right)+c$

$= \frac{2}{35}(5u+42)u^{\frac{5}{2}}+c$

$= \frac{2}{35}[5(x-4)+42](x-4)^{\frac{5}{2}}+c$

$= \frac{2(5x-20+42)(x-4)^{\frac{5}{2}}}{35}+c$

$= \frac{2(5x+22)(x-4)^{\frac{5}{2}}}{35}+c$

n i $\int (x-3)(2x+1)^{\frac{5}{2}}\,dx$

Let $u = 2x+1$ and $x = \frac{u-1}{2}$

$\frac{du}{dx} = 2$ or $dx = \frac{du}{2}$

So $\int (x-3)(2x+1)^{\frac{5}{2}}\,dx$

$= \int \left(\frac{u-1}{2}-3\right)(u)^{\frac{5}{2}}\frac{du}{2}$

$= \int \frac{u^{\frac{5}{2}}}{2}\left(\frac{u-1-6}{2}\right)du$

$= \int \frac{u^{\frac{5}{2}}}{4}(u-7)\,du$

$= \int \frac{u^{\frac{7}{2}}-7u^{\frac{5}{2}}}{4}du$

ii $= \frac{1}{4}\int \left(u^{\frac{7}{2}}-7u^{\frac{5}{2}}\right)du$

$= \frac{1}{4}\left(\frac{2}{9}u^{\frac{9}{2}}-\frac{7\times 2}{7}u^{\frac{7}{2}}\right)+c$

$= \frac{1}{2}\left(\frac{u^{\frac{9}{2}}}{9}-u^{\frac{7}{2}}\right)+c$

$= \frac{1}{18}\left(u^{\frac{9}{2}}-9u^{\frac{7}{2}}\right)+c$

$= \frac{1}{18}(u-9)u^{\frac{7}{2}}+c$

$= \frac{(2x+1-9)(2x+1)^{\frac{7}{2}}}{18}+c$

$= \frac{(2x-8)(2x+1)^{\frac{7}{2}}}{18}+c$

$= \frac{2(x-4)(2x+1)^{\frac{7}{2}}}{18}+c$

$= \frac{(x-4)(2x+1)^{\frac{7}{2}}}{9}+c$

o i $\int \frac{2x}{\sqrt{x-6}}dx$

Let $u = x-6$ and $x = u+6$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int \frac{2x}{\sqrt{x-6}}dx$

$= \int \frac{2(u+6)}{\sqrt{u}}du$

$= \int \frac{2u+12}{u^{\frac{1}{2}}} du$

$= \int \left(2u^{\frac{1}{2}} + 12u^{\frac{-1}{2}}\right) du$

ii $= \frac{4}{3}u^{\frac{3}{2}} + 24u^{\frac{1}{2}} + c$

$= \frac{4}{3}(u + 3 \times 6)u^{\frac{1}{2}} + c$

$= \frac{4}{3}(u + 18)u^{\frac{1}{2}} + c$

$= \frac{4}{3}(x - 6 + 18)(x - 6)^{\frac{1}{2}} + c$

$= \frac{4(x+12)(x-6)^{\frac{1}{2}}}{3} + c$

p i $\int \frac{3x}{\sqrt{8-x}} dx$

Let $u = 8 - x$ and $x = 8 - u$

$\frac{du}{dx} = -1$ or $dx = -du$

So $\int \frac{3x}{\sqrt{8-x}} dx$

$= \int \frac{3(8-u)}{\sqrt{u}} \times -du$

$= \int \frac{3(u-8)}{u^{\frac{1}{2}}} du$

$= \int \left(3u^{\frac{1}{2}} - 24u^{\frac{-1}{2}}\right) du$

ii $= 2u^{\frac{3}{2}} - 48u^{\frac{1}{2}} + c$

$= 2(u - 24)u^{\frac{1}{2}} + c$

$= 2(8 - x - 24)(8 - x)^{\frac{1}{2}} + c$

$= -2(x + 16)(8 - x)^{\frac{1}{2}} + c$

2 a C

b Let $u = x + 2$ and $x = u - 2$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int 4x\sqrt{x+2}\, dx$

$= \int 4(u-2)\sqrt{u}\, du$

$= \int 4u^{\frac{1}{2}}(u - 2)\, du$

$= \int (4u^{\frac{3}{2}} - 8u^{\frac{1}{2}})\, du$

$\therefore$ **E**

3 a $\int \frac{x^2}{\sqrt{x-1}} dx$

Let $u = x - 1$ and $x = u + 1$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int \frac{x^2}{\sqrt{x-1}} dx$

$= \int \frac{(u+1)^2}{\sqrt{u}} du$

$= \int \frac{(u^2 + 2u + 1)}{u^{\frac{1}{2}}} du$

$= \int \left(u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + u^{\frac{-1}{2}}\right) du$

$\therefore$ **B**

b $\int \left(u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + u^{\frac{-1}{2}}\right) du$

$= \frac{2}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + c$

$= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + c$

$\therefore$ **D**

4 a $\int x^2(x-4)^4 dx$

Let $u = x - 4$ and $x = u + 4$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int x^2(x-4)^4 dx$

$= \int (u+4)^2(u)^4 du$

$= \int u^4(u^2 + 8u + 16) du$

$= \int (u^6 + 8u^5 + 16u^4) du$

$= \frac{1}{7}u^7 + \frac{8}{6}u^6 + \frac{16}{5}u^5 + c$

$= \frac{1}{7}(x-4)^7 + \frac{4}{3}(x-4)^6 + \frac{16}{5}(x-4)^5 + c$

b $\int x^2(5-x)^3\, dx$

Let $u = 5 - x$ and $x = 5 - u$

$\frac{du}{dx} = -1$ or $dx = -du$

So $\int x^2(5-x)^3\, dx$

$= \int (5-u)^2(u)^3 \times -du$

$= -\int u^3(u-5)^2 du$

$= -\int u^3(u^2 - 10u + 25) du$

$= -\int (u^5 - 10u^4 + 25u^3) du$

$= \int (10u^4 - 25u^3 - u^5) du$

$= \frac{10}{5}u^5 - \frac{25}{4}u^4 - \frac{1}{6}u^6 + c$

$= 2(5-x)^5 - \frac{25}{4}(5-x)^4 - \frac{1}{6}(5-x)^6 + c$

c $\int x^2\sqrt{x-1}\, dx$

Let $u = x - 1$ and $x = u + 1$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int x^2\sqrt{x-1}\, dx$

$= \int (u+1)^2\sqrt{u}\, du$

$= \int (u^2 + 2u + 1)u^{\frac{1}{2}}\, du$

$= \int \left(u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

$= \frac{2}{7}u^{\frac{7}{2}} + \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{7}(x-1)^{\frac{7}{2}} + \frac{4}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + c$

d $\int x^2\sqrt{3-x}\,dx$

Let $u = 3 - x$ and $x = 3 - u$

$\frac{du}{dx} = -1$ or $dx = -du$

So $\int x^2\sqrt{3-x}\,dx$

$= \int (3-u)^2\sqrt{u} \times -du$

$= -\int (9 - 6u + u^2)u^{\frac{1}{2}}du$

$= -\int \left(9u^{\frac{1}{2}} - 6u^{\frac{3}{2}} + u^{\frac{5}{2}}\right)du$

$= \int \left(-9u^{\frac{1}{2}} + 6u^{\frac{3}{2}} - u^{\frac{5}{2}}\right)du$

$= \frac{-9\times 2}{3}u^{\frac{3}{2}} + \frac{6\times 2}{5}u^{\frac{5}{2}} - \frac{2}{7}u^{\frac{7}{2}} + c$

$= -6(3-x)^{\frac{3}{2}} + \frac{12}{5}(3-x)^{\frac{5}{2}} - \frac{2}{7}(3-x)^{\frac{7}{2}} + c$

e $\int x^2(x+2)^{\frac{4}{3}}dx$

Let $u = x + 2$ and $x = u - 2$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int x^2(x+2)^{\frac{4}{3}}dx$

$= \int (u-2)^2(u)^{\frac{4}{3}}du$

$= \int (u^2 - 4u + 4)u^{\frac{4}{3}}du$

$= \int \left(u^{\frac{10}{3}} - 4u^{\frac{7}{3}} + 4u^{\frac{4}{3}}\right)du$

$= \frac{3}{13}u^{\frac{13}{3}} - \frac{12}{10}u^{\frac{10}{3}} + \frac{12}{7}u^{\frac{7}{3}} + c$

$= \frac{3}{13}(x+2)^{\frac{13}{3}} - \frac{6}{5}(x+2)^{\frac{10}{3}} + \frac{12}{7}(x+2)^{\frac{7}{3}} + c$

f $\int x^2(1-x)^{\frac{3}{4}}dx$

Let $u = 1 - x$ and $x = 1 - u$

$\frac{du}{dx} = -1$ or $dx = -du$

So $\int x^2(1-x)^{\frac{3}{4}}dx$

$= \int (1-u)^2(u)^{\frac{3}{4}} \times -du$

$= -\int (1 - 2u + u^2)u^{\frac{3}{4}}du$

$= \int \left(-u^{\frac{3}{4}} + 2u^{\frac{7}{4}} - u^{\frac{11}{4}}\right)du$

$= -\frac{4}{7}u^{\frac{7}{4}} + \frac{8}{11}u^{\frac{11}{4}} - \frac{4}{15}u^{\frac{15}{4}} + c$

$= -\frac{4}{7}(1-x)^{\frac{7}{4}} + \frac{8}{11}(1-x)^{\frac{11}{4}} - \frac{4}{15}(1-x)^{\frac{15}{44}} + c$

g $\int (x+1)^2\sqrt{x-2}\,dx$

Let $u = x - 2$ and $x = u + 2$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int (x+1)^2\sqrt{x-2}\,dx$

$= \int (u+2+1)^2\sqrt{u}\,du$

$= \int (u+3)^2 u^{\frac{1}{2}}du$

$= \int (u^2 + 6u + 9)u^{\frac{1}{2}}du$

$= \int \left(u^{\frac{5}{2}} + 6u^{\frac{3}{2}} + 9u^{\frac{1}{2}}\right)du$

$= \frac{2}{7}u^{\frac{7}{2}} + \frac{12}{5}u^{\frac{5}{2}} + \frac{18}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{7}(x-2)^{\frac{7}{2}} + \frac{12}{5}(x-2)^{\frac{5}{2}} + 6(x-2)^{\frac{3}{2}} + c$

h $\int (x-3)^2\sqrt{x+1}\,dx$

Let $u = x + 1$ and $x = u - 1$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int (x-3)^2\sqrt{x+1}\,dx$

$= \int (u-1-3)^2\sqrt{u}\,du$

$= \int (u-4)^2 u^{\frac{1}{2}}du$

$= \int (u^2 - 8u + 16)u^{\frac{1}{2}}du$

$= \int \left(u^{\frac{5}{2}} - 8u^{\frac{3}{2}} + 16u^{\frac{1}{2}}\right)du$

$= \frac{2}{7}u^{\frac{7}{2}} - \frac{16}{5}u^{\frac{5}{2}} + \frac{32}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{7}(x+1)^{\frac{7}{2}} - \frac{16}{5}(x+1)^{\frac{5}{2}} + \frac{32}{3}(x+1)^{\frac{3}{2}} + c$

i $\int \frac{e^x}{e^x+1}dx$

Let $u = e^x + 1$

$\frac{du}{dx} = e^x$ or $dx = \frac{du}{e^x}$

So $\int \frac{e^x}{e^x+1}dx$

$= \int \frac{e^x}{u}\frac{du}{e^x}$

$= \int \frac{1}{u}du$

$= \log_e(u) + c$

$= \log_e(e^x + 1) + c$

j $\int \frac{x^2}{\sqrt{x+1}}dx$

Let $u = x + 1$ and $x = u - 1$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int \frac{x^2}{\sqrt{x+1}}dx$

$= \int \frac{(u-1)^2}{\sqrt{u}}du$

$= \int u^{\frac{-1}{2}}(u^2 - 2u + 1)\,du$

$= \int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + u^{\frac{-1}{2}}\right) du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + c$

$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{4}{3}(x+1)^{\frac{3}{2}} + 2(x+1)^{\frac{1}{2}} + c$

k $\int \frac{2x^2}{\sqrt{3-x}}\,dx$

Let $u = 3 - x$ and $x = 3 - u$

$\frac{du}{dx} = -1$ or $dx = -du$

So $\int \frac{2x^2}{\sqrt{3-x}}\,dx$

$= \int \frac{2(3-u)^2}{\sqrt{u}} \times -du$

$= -2\int u^{\frac{-1}{2}}(9 - 6u + u^2)\,du$

$= -2\int \left(9u^{\frac{-1}{2}} - 6u^{\frac{1}{2}} + u^{\frac{3}{2}}\right) du$

$= -2\left(18u^{\frac{1}{2}} - \frac{12}{3}u^{\frac{3}{2}} + \frac{2}{5}u^{\frac{5}{2}}\right) + c$

$= -36(3-x)^{\frac{1}{2}} + 8(3-x)^{\frac{3}{2}} - \frac{4}{5}(3-x)^{\frac{5}{2}} + c$

l $\int x^3\sqrt{x-1}\,dx$

Let $u = x - 1$ and $x = u + 1$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int x^3\sqrt{x-1}\,dx$

$= \int (u+1)^3\sqrt{u}\,du$

$= \int (u^3 + 3u^3 + 3u + 1)u^{\frac{1}{2}}\,du$

$= \int \left(u^{\frac{7}{2}} + 3u^{\frac{5}{2}} + 3u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

$= \frac{2}{9}u^{\frac{9}{2}} + \frac{6}{7}u^{\frac{7}{2}} + \frac{6}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{9}(x-1)^{\frac{9}{2}} + \frac{6}{7}(x-1)^{\frac{7}{2}} + \frac{6}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + c$

m $\int \frac{x^3}{x+4}\,dx$

Let $u = x + 4$ and $x = u - 4$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int \frac{x^3}{\sqrt{x+4}}\,dx$

$= \int \frac{(u-4)^3}{\sqrt{u}}\,du$

$= \int u^{\frac{-1}{2}}(u^3 - 3\times 4u^2 + 3\times 4^2 u - 4^3)\,du$

$= \int u^{\frac{-1}{2}}(u^3 - 12u^2 + 48u - 64)\,du$

$= \int \left(u^{\frac{5}{2}} - 12u^{\frac{3}{2}} + 48u^{\frac{1}{2}} - 64u^{\frac{-1}{2}}\right) du$

$= \frac{2}{7}u^{\frac{7}{2}} - \frac{24}{5}u^{\frac{5}{2}} + \frac{96}{3}u^{\frac{3}{2}} - 128u^{\frac{1}{2}} + c$

$= \frac{2}{7}(x+4)^{\frac{7}{2}} - \frac{24}{5}(x+4)^{\frac{5}{2}} + 32(x+4)^{\frac{3}{2}} - 128(x+4)^{\frac{1}{2}} + c$

n $\int \frac{2x^3}{\sqrt{1-x}}\,dx$

Let $u = 1 - x$ and $x = 1 - u$

$\frac{du}{dx} = -1$ or $dx = -du$

So $\int \frac{2x^3}{\sqrt{1-x}}\,dx$

$= \int \frac{2(1-u)^3}{\sqrt{u}} \times -du$

$= 2\int u^{\frac{-1}{2}}(1 - 3u + 3u^2 - u^3)\,du$

$= 2\int \left(u^{\frac{5}{2}} - 3u^{\frac{3}{2}} + 3u^{\frac{1}{2}} - u^{\frac{-1}{2}}\right) du$

$= 2\left(\frac{2}{7}u^{\frac{7}{2}} - \frac{6}{5}u^{\frac{5}{2}} + 2u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) + c$

$= \frac{4}{7}(1-x)^{\frac{7}{2}} - \frac{12}{5}(1-x)^{\frac{5}{2}} + 4(1-x)^{\frac{3}{2}} - 4(1-x)^{\frac{1}{2}} + c$

o $\int \frac{x+3}{(x-2)^2}\,dx$

Let $u = x - 2$ and $x = u + 2$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int \frac{x+3}{(x-2)^2}\,dx$

$= \int \frac{u+2+3}{(u)^2}\,du$

$= \int u^{-2}(u+5)\,du$

$= \int (u^{-1} + 5u^{-2})\,du$

$= \log_e(u) - 5u^{-1} + c$

$= \log_e(x-2) - \frac{5}{x-2} + c$

p $\int \frac{2x-1}{(x-1)^3}\,dx$

Let $u = x + 1$ and $x = u - 1$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int \frac{2x-1}{(x+1)^3}\,dx$

$= \int \frac{2(u-1)-1}{u^3}\,du$

$= \int \frac{2u-2-1}{u^3}\,du$

$= \int u^{-3}(2u-3)\,du$

$= \int (2u^{-2} - 3u^{-3})\,du$

$= -2u^{-1} - \frac{3}{-2}u^{-2} + c$

$= \frac{3}{2u^2} - \frac{2}{u} + c$

$= \frac{3}{2(x+1)^2} - \frac{2}{x+1} + c$

q $\int \frac{4x}{(x+2)^2} dx$

Let $u = x + 2$ and $x = u - 2$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int \frac{4x}{(x+2)^2} dx$

$= \int \frac{4(u-2)}{(u)^2} du$

$= 4\int (u^{-1} - 2u^{-2}) du$

$= 4 (\log_e(u) + 2 u^{-1}) + c$

$= 4\log_e(u) + \frac{8}{u} + c$

$= 4\log_e(x+2) + \frac{8}{x+2} + c$

r $\int \frac{x^2}{(x-1)^2} dx$

Let $u = x - 1$ and $x = u + 1$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int \frac{x^2}{(x-1)^2} dx$

$= \int \frac{(u+1)^2}{(u)^2} du$

$= \int \frac{u^2 + 2u + 1}{u^2} du$

$= \int (1 + 2u^{-1} + u^{-2}) du$

$= u + 2\log_e(u) - u^{-1} + c$

$= (x-1) + 2\log_e(x-1) - \frac{1}{x-1} + c$

s $\int \frac{(x+3)^2}{\sqrt{x+2}} dx$

Let $u = x + 2$ and $x = u - 2$

$\frac{du}{dx} = 1$ or $dx = du$

$\int \frac{(x+3)^2}{\sqrt{x+2}} dx$

$= \int \frac{(u-2+3)^2}{\sqrt{u}} du$

$= \int u^{\frac{-1}{2}} (u+1)^2 du$

$= \int u^{\frac{-1}{2}} (u^2 + 2u + 1) du$

$= \int \left(u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + u^{\frac{-1}{2}} \right) du$

$= \frac{2}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + 2u^{\frac{-1}{2}} + c$

$= \frac{2}{5}(x+2)^{\frac{5}{2}} + \frac{4}{3}(x+2)^{\frac{3}{2}} + 2(x+2)^{\frac{1}{2}} + c$

t $\int \frac{(x-2)^2}{\sqrt{2-x}} dx$

Let $u = 2 - x$ and $x = 2 - u$

$\frac{du}{dx} = -1$ or $dx = -du$

So $\int \frac{(x-2)^2}{\sqrt{2-x}} dx$

$= \int \frac{(2-u-2)^2}{\sqrt{u}} \times -du$

$= -\int \frac{(-u)^2}{\sqrt{u}} du$

$= -\int u^{\frac{-1}{2}} u^2 du$

$= -\int u^{\frac{3}{2}} du$

$= -\frac{2}{5}u^{\frac{5}{2}} + c$

$= -\frac{2}{5}(2-x)^{\frac{5}{2}} + c$

u $\int \frac{e^{2x}}{e^x + 2} dx$

Let $u = e^x + 2$ and $e^x = u - 2$

$\frac{du}{dx} = e^x$ or $dx = \frac{du}{e^x}$

So $\int \frac{e^{2x}}{e^x + 2} dx$

$= \int \frac{e^{2x}}{u} \times \frac{du}{e^x}$

$= \int \frac{e^x}{u} du$

$= \int \frac{u-2}{u} du$

$= \int \left(1 - \frac{2}{u}\right) du$

$= u - 2 \log_e(u) + c$

$= e^x + 2 - 2\log_e(e^x + 2) + c$

v $\int \frac{e^{3x}}{e^x - 1} dx$

Let $u = e^x - 1$ and $e^x = u + 1$

$\frac{du}{dx} = e^x$ or $dx = \frac{du}{e^x}$

So $\int \frac{e^{3x}}{e^x - 1} dx$

$= \int \frac{e^{3x}}{u} \times \frac{du}{e^x}$

$= \int \frac{(e^x)^2}{u} du$

$= \int \frac{(u+1)^2}{u} du$

$= \int \frac{(u^2 + 2u + 1)}{u} du$

$= \int (u + 2 + \frac{1}{u}) du$

$= \frac{1}{2}u^2 + 2u + \log_e(u) + c$

$= \frac{1}{2}(e^x - 1)^2 + 2(e^x - 1) + \log_e(e^x - 1) + c$

5 a If $f'(x) = -(5-x)^{\frac{1}{2}} + 10(5-x)^{\frac{-1}{2}}$ and $f(1) = -2$, find $f(x)$

$f'(x) = -(5-x)^{\frac{1}{2}} + 10(5-x)^{\frac{-1}{2}}$

$f(x) = \int\left[-(5-x)^{\frac{1}{2}} + 10(5-x)^{\frac{-1}{2}}\right]dx$

Let $u = 5 - x$

$\frac{du}{dx} = -1$ or $dx = -du$

So $f(x) = \int\left[-(u)^{\frac{1}{2}} + 10(u)^{\frac{-1}{2}}\right]\times -du$

$= \int\left(u^{\frac{1}{2}} - 10u^{\frac{-1}{2}}\right)du$

$= \frac{2}{3}u^{\frac{3}{2}} - 20u^{\frac{1}{2}} + c$

$= \frac{2}{3}(5-x)^{\frac{3}{2}} - 20(5-x)^{\frac{1}{2}} + c$

$f(1) = \frac{2}{3}(5-1)^{\frac{3}{2}} - 20(5-1)^{\frac{1}{2}} + c = -2$

$\frac{2}{3}(\sqrt{4})^3 - 20\sqrt{4} + c = -2$

$\frac{2}{3}\times 2^3 - 40 + c = -2$

$\frac{16}{3} - 40 + c = -2$

$c = 38 - \frac{16}{3}$

$c = \frac{114-16}{3}$

$c = \frac{98}{3}$

$c = 32\frac{2}{3}$

Therefore $f(x) = \frac{2}{3}(5-x)^{\frac{3}{2}} - 20(5-x)^{\frac{1}{2}} + 32\frac{2}{3}$

b For $(5-x)^{\frac{3}{2}}$ and $(5-x)^{\frac{1}{2}}$ to exist $5 - x \geq 0$, so $x \leq 5$
Therefore the domain of $f(x)$ is $x \leq 5$.

6 a If $f'(x) = \frac{5(x+1)^{\frac{3}{2}}}{2} - 3(x+1)^{\frac{1}{2}} + \frac{(x+1)^{\frac{-1}{2}}}{2}$ and $f(0) = 1$, find $f(x)$.

$f'(x) = \frac{5(x+1)^{\frac{3}{2}}}{2} - 3(x+1)^{\frac{1}{2}} + \frac{(x+1)^{\frac{-1}{2}}}{2}$

$f(x) = \int\left[\frac{5(x+1)^{\frac{3}{2}}}{2} - 3(x+1)^{\frac{1}{2}} + \frac{(x+1)^{\frac{-1}{2}}}{2}\right]dx$

Let $u = x + 1$

$\frac{du}{dx} = 1$ or $dx = du$

So $f(x) = \int\left[\frac{5(u)^{\frac{3}{2}}}{2} - 3(u)^{\frac{1}{2}} + \frac{(u)^{\frac{-1}{2}}}{2}\right]du$

$= u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}} + c$

$f(x) = (x+1)^{\frac{5}{2}} - 2(x+1)^{\frac{3}{2}} + (x+1)^{\frac{1}{2}} + c$

$f(0) = (0+1)^{\frac{5}{2}} - 2(0+1)^{\frac{3}{2}} + (0+1)^{\frac{1}{2}} + c = 1$

$1 - 2 + 1 + c = 1$

$c = 1$

Therefore $f(x) = (x+1)^{\frac{5}{2}} - 2(x+1)^{\frac{3}{2}} + (x+1)^{\frac{1}{2}} + 1$

b For $(x+1)^{\frac{5}{2}}$, $(x+1)^{\frac{3}{2}}$ and $(x+1)^{\frac{1}{2}}$ to exist $x + 1 \geq 0$
so $x \geq -1$
Therefore the domain of $f(x)$ is $x \geq -1$

7 a Given that $g'(x) = \frac{2x+1}{(x-1)^2}$ and $g(2) = 0$, find $g(x)$

$g'(x) = \frac{2x+1}{(x-1)^2}$

$g(x) = \int\frac{2x+1}{(x-1)^2}dx$

Let $u = x - 1$ and $x = u + 1$

$\frac{du}{dx} = 1$ or $dx = du$

So $g(x) = \int\frac{2(u+1)+1}{(u)^2}du$

$= \int\frac{2u+2+1}{u^2}du$

$= \int\frac{2u+3}{u^2}du$

$= \int(2u^{-1} + 3u^{-2})du$

$= 2\log_e(u) - 3u^{-1} + c$

$g(x) = 2\log_e(x-1) - \frac{3}{x-1} + c$

$g(2) = 2\log_e(2-1) - \frac{3}{2-1} + c = 0$

$2\log_e(1) - 3 + c = 0$

$c = 3$

So $g(x) = 2\log_e(x-1) - \frac{3}{x-1} + 3$

b For $\log_e(x-1)$ to exist, $x - 1 > 0$ so $x > 1$, and for $\frac{3}{x-1}$ to exist $x - 1 \neq 0$ so $x \neq 1$.
Therefore the domain of $g(x)$ is $x > 1$.

8 a Given that $g(0) = 2 - \log_e(2)$ and $g'(x) = \frac{e^{2x}}{e^x+1}$, find $g(x)$

$g'(x) = \frac{e^{2x}}{e^x+1}$

$g(x) = \int\frac{e^{2x}}{e^x+1}dx$

Let $u = e^x + 1$ and $e^x = u - 1$

$\frac{du}{dx} = e^x$ or $dx = \frac{du}{dx}$

So $g(x) = \int\frac{e^{2x}}{u}\times\frac{du}{e^x}$

$= \int\frac{e^x}{u}du$

$= \int\frac{u-1}{u}du$

$= \int\left(1 - \frac{1}{u}\right)du$

$= u - \log_e(u) + c$

$g(x) = e^x + 1 - \log_e(e^x + 1) + c$

$g(0) = e^0 + 1 - \log_e(e^0 + 1) + c = 2 - \log_e(2)$

$1 + 1 - \log_e(1 + 1)\ c = 2 - \log_e(2)$

$2 - \log_e(2) + c = 2 - \log_e(2)$

$c = 0$

Therefore $g(x) = e^x + 1 - \log_e(e^x + 1)$

b For $\log_e(e^x + 1)$ to exist $e^x + 1 > 0$, which is true for all x. Therefore the domain of $g(x)$ is R.

Exercise 4D — Antiderivatives involving trigonometric identities

1 a $\int \cos^2(x)\,dx$

$= \int \frac{1}{2}(1+\cos(2x))\,dx$

$= \frac{1}{2}\int (1+\cos(2x))\,dx$

$= \frac{1}{2}(x + \frac{1}{2}\sin(2x)) + c$

$= \frac{1}{2}x + \frac{1}{4}\sin(2x) + c$

b $\int \sin^2(2x)\,dx$

$= \int \frac{1}{2}(1-\cos(4x))\,dx$

$= \frac{1}{2}\int (1-\cos(4x))\,dx$

$= \frac{1}{2}(x - \frac{1}{4}\sin(4x)) + c$

$= \frac{1}{2}x - \frac{1}{8}\sin(4x) + c$

c $\int 2\cos^2(4x)\,dx$

$= \int 2\left(\frac{1}{2}\right)(1+\cos(8x))\,dx$

$= \int (1+\cos(8x))\,dx$

$= x + \frac{1}{8}\sin(8x) + c$

d $\int 4\sin^2(3x)\,dx$

$= \int 4\left(\frac{1}{2}\right)(1-\cos(6x))\,dx$

$= 2\int (1-\cos(6x))\,dx$

$= 2(x - \frac{1}{6}\sin(6x)) + c$

$= 2x - \frac{1}{3}\sin(6x) + c$

e $\int \cos^2(5x)\,dx$

$= \int \frac{1}{2}(1+\cos(10x))\,dx$

$= \frac{1}{2}\int (1+\cos(10x))\,dx$

$= \frac{1}{2}(x + \frac{1}{10}\sin(10x)) + c$

$= \frac{1}{2}x + \frac{1}{20}\sin(10x) + c$

f $\int \sin^2(6x)\,dx$

$= \int \frac{1}{2}(1-\cos(12x))\,dx$

$= \frac{1}{2}\int (1-\cos(12x))\,dx$

$= \frac{1}{2}(x - \frac{1}{12}\sin(12x)) + c$

$= \frac{1}{2}x - \frac{1}{24}\sin(12x) + c$

g $\int \cos^2\left(\frac{x}{2}\right)dx$

$= \int \frac{1}{2}(1+\cos(x))\,dx$

$= \frac{1}{2}\int (1+\cos(x))\,dx$

$= \frac{1}{2}(x + \sin(x)) + c$

h $\int \sin^2\left(\frac{x}{3}\right)dx$

$= \int \frac{1}{2}\left(1-\cos\left(\frac{2x}{3}\right)\right)dx$

$= \frac{1}{2}\int \left(1-\cos\left(\frac{2x}{3}\right)\right)dx$

$= \frac{1}{2}\left(x-\frac{3}{2}\sin\left(\frac{2x}{3}\right)\right)+c$

$= \frac{1}{2}x-\frac{3}{4}\sin\left(\frac{2x}{3}\right)+c$

i $\int 3\cos^2\left(\frac{x}{6}\right)dx$

$= \int 3\left(\frac{1}{2}\right)\left(1+\cos\left(\frac{x}{3}\right)\right)dx$

$= \frac{3}{2}\int \left(1+\cos\left(\frac{x}{3}\right)\right)dx$

$= \frac{3}{2}\left(x+3\sin\left(\frac{x}{3}\right)\right)+c$

$= \frac{3}{2}x+\frac{9}{2}\sin\left(\frac{x}{3}\right)+c$

j $\int 2\sin^2\left(\frac{x}{4}\right)dx$

$= \int 2\left(\frac{1}{2}\right)\left(1-\cos\left(\frac{x}{2}\right)\right)dx$

$= \int \left(1-\cos\left(\frac{x}{2}\right)\right)dx$

$= x-2\sin\left(\frac{x}{2}\right)+c$

k $\int \cos^2\left(\frac{2x}{3}\right)dx$

$= \int \frac{1}{2}\left(1+\cos\left(\frac{4x}{3}\right)\right)dx$

$= \frac{1}{2}\int \left(1+\cos\left(\frac{4x}{3}\right)\right)dx$

$= \frac{1}{2}\left(x+\frac{3}{4}\sin\left(\frac{4x}{3}\right)\right)+c$

$= \frac{1}{2}x+\frac{3}{8}\sin\left(\frac{4x}{3}\right)+c$

l $\int \sin^2\left(\frac{3x}{2}\right)dx$

$= \int \frac{1}{2}(1-\cos(3x))\,dx$

$= \frac{1}{2}\int (1-\cos(3x))\,dx$

$= \frac{1}{2}\,(x - \frac{1}{3}\sin(3x)) + c$

$= \frac{1}{2}x - \frac{1}{6}\sin(3x) + c$

2 a $\int 2\sin(x)\cos(x)\,dx$

$= \int \sin(2x)\,dx$

$= \frac{-1}{2}\cos(2x) + c$

$= \frac{-1}{2}(1 - 2\sin^2(x)) + c$

$= \sin^2(x) - \frac{1}{2} + c$ or $\sin^2(x) + c$ because c is any constant.

b $\int 4\sin(2x)\cos(2x)\,dx$

$= \int 2\sin(4x)\,dx$

$= 2\int \sin(4x)\,dx$

$= 2\left(\frac{-1}{4}\cos(4x)\right) + c$

$= \frac{-1}{2}\cos(4x) + c$

c $\int \sin(3x)\cos(3x)\,dx$

$= \frac{1}{2}\int 2\sin(3x)\cos(3x)\,dx$

$= \frac{1}{2}\int \sin(6x)\,dx$

$= \frac{1}{2}\left(\frac{-1}{6}\cos(6x)\right) + c$

$= \frac{-1}{12}\cos(6x) + c$

d $\int -2\sin(4x)\cos(4x)\,dx$

$= -\int \sin(8x)\,dx$

$= -\left(\frac{-1}{8}\cos(8x)\right) + c$

$= \frac{1}{8}\cos(8x) + c$

e $\int \sin^2(x)\cos^2(x)\,dx$

$= \frac{1}{4}\int (2\sin(x)\cos(x))^2\,dx$

$= \frac{1}{4}\int \sin^2(2x)\,dx$

$= \frac{1}{4}\int \frac{1}{2}(1-\cos(4x))\,dx$

$= \frac{1}{8}\int (1-\cos(4x))\,dx$

$= \frac{1}{8}\,(x - \frac{1}{4}\sin(4x)) + c$

$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + c$

f $\int \sin^2(2x)\cos^2(2x)\,dx$

$= \frac{1}{4}\int (2\sin(2x)\cos(2x))\,dx$

$= \frac{1}{4}\int \sin^2(4x)\,dx$

$= \frac{1}{4}\int \frac{1}{2}(1-\cos(8x))\,dx$

$= \frac{1}{8}\int (1-\cos(8x))\,dx$

$= \frac{1}{8}\,(x - \frac{1}{8}\sin(8x)) + c$

$= \frac{1}{8}x - \frac{1}{64}\sin(8x) + c$

g $\int 2\sin^2(4x)\cos^2(4x)\,dx$

$= \frac{1}{2}\int (2\sin(4x)\cos(4x))^2\,dx$

$= \frac{1}{2}\int \sin^2(8x)\,dx$

$= \frac{1}{2}\int \frac{1}{2}(1-\cos(16x))\,dx$

$= \frac{1}{4}\int (1-\cos(16x))\,dx$

$= \frac{1}{4}\,(x - \frac{1}{16}\sin(16x)) + c$

$= \frac{1}{4}x - \frac{1}{64}\sin(16x) + c$

h $\int 2\sin^2(3x)\cos^2(3x)\,dx$

$= \frac{1}{2}\int (2\sin(3x)\cos(3x))^2\,dx$

$= \frac{1}{2}\int \sin^2(6x)\,dx$

$= \frac{1}{2}\int \frac{1}{2}(1-\cos(12x))\,dx$

$= \frac{1}{4}\int (1-\cos(12x))\,dx$

$= \frac{1}{4}\,(x - \frac{1}{12}\sin(12x)) + c$

$= \frac{1}{4}x - \frac{1}{48}\sin(12x) + c$

i $\int 6\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)dx$

$= \frac{6}{4}\int \left(2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)^2 dx$

$= \frac{3}{2}\int \sin^2(x)\,dx$

$= \frac{3}{2}\int \frac{1}{2}(1-\cos(2x))\,dx$

$= \frac{3}{4}\int (1-\cos(2x))\,dx$

$= \frac{3}{4}\,(x - \frac{1}{2}\sin(2x)) + c$

$= \frac{3}{4}x - \frac{3}{8}\sin(2x)) + c$

j $\int 4\sin^2\left(\frac{x}{3}\right)\cos^2\left(\frac{x}{3}\right)dx$

$= \int \left(2\sin\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right)\right)^2 dx$

$= \int \sin^2\left(\frac{2x}{3}\right)dx$

$= \int \frac{1}{2}\left(1-\cos\left(\frac{4x}{3}\right)\right)dx$

$= \frac{1}{2}\int\left(1-\cos\left(\frac{4x}{3}\right)\right)dx$

$= \frac{1}{2}\left(x-\frac{3}{4}\sin\left(\frac{4x}{3}\right)\right)+c$

$= \frac{1}{2}x-\frac{3}{8}\sin\left(\frac{4x}{3}\right)+c$

k $\int \sin^2\left(\frac{5x}{2}\right)\cos^2\left(\frac{5x}{2}\right)dx$

$= \frac{1}{4}\int\left(2\sin\left(\frac{5x}{2}\right)\cos\left(\frac{5x}{2}\right)\right)^2 dx$

$= \frac{1}{4}\int \sin^2(5x)\,dx$

$= \frac{1}{4}\int\frac{1}{2}(1-\cos(10x))\,dx$

$= \frac{1}{8}\int(1-\cos(10x))\,dx$

$= \frac{1}{8}\left(x-\frac{1}{10}\sin(10x)\right)+c$

$= \frac{1}{8}x-\frac{1}{80}\sin(10x)+c$

l $\int -2\sin^2\left(\frac{4x}{3}\right)\cos^2\left(\frac{4x}{3}\right)dx$

$= \frac{-1}{2}\int\left(2\sin\left(\frac{4x}{3}\right)\cos\left(\frac{4x}{3}\right)\right)^2 dx$

$= \frac{-1}{2}\int\sin^2\left(\frac{8x}{3}\right)dx$

$= \frac{-1}{2}\int\frac{1}{2}\left(1-\cos\left(\frac{16x}{3}\right)\right)dx$

$= \frac{-1}{4}\int\left(1-\cos\left(\frac{16x}{3}\right)\right)dx$

$= \frac{-1}{4}\left(x-\frac{3}{16}\sin\left(\frac{16x}{3}\right)\right)+c$

$= \frac{-1}{4}x+\frac{3}{64}\sin\left(\frac{16x}{3}\right)+c$

3 a $\int \sin^2(ax)\,dx$

$= \int\frac{1}{2}(1-\cos(2ax))\,dx$

$= \frac{1}{2}\int(1-\cos(2ax))\,dx$

$= \frac{1}{2}\left(x-\frac{1}{2a}\sin(2ax)\right)+c$

$= \frac{x}{2}-\frac{\sin(2ax)}{4a}+c$

$\therefore$ **C**

b $\int \sin^2(ax)\cos^2(ax)\,dx$

$= \frac{1}{4}\int(2\sin(ax)\cos(ax))^2\,dx$

$= \frac{1}{4}\int\sin^2(2ax)\,dx$

$= \frac{1}{4}\int\frac{1}{2}(1-\cos(4ax))\,dx$

$= \frac{1}{8}\int(1-\cos(4ax))\,dx$

$= \frac{1}{8}\left(x-\frac{1}{4a}\sin(4ax)\right)+c$

$= \frac{x}{8}-\frac{\sin(4ax)}{32a}+c$

$\therefore$ **A**

c $\int \cos^3(ax)\,dx$

$= \int(1-\sin^2(ax))\cos(ax)\,dx$

Let $u=\sin(ax)$

$\frac{du}{dx}=a\cos(ax)$ or $dx=\frac{du}{a\cos(ax)}$

So $\int(1-\sin^2(ax))\cos(ax)\,dx$

$= \int(1-u^2)\cos(ax)\frac{du}{a\cos(ax)}$

$= \frac{1}{a}\int(1-u^2)\,du$

$= \frac{1}{a}\left(u-\frac{1}{3}u^3\right)+c$

$= \frac{1}{a}\left(\sin(ax)-\frac{1}{3}\sin^3(ax)\right)+c$

$= \frac{1}{3a}(3\sin(ax)-\sin^3(ax))+c$

$\therefore$ **E**

4 a $\int \sin^3(x)\,dx$

$\int(1-\cos^2(x))\sin(x)\,dx$

Let $u=\cos(x)$

$\frac{du}{dx}=-\sin(x)$ or $dx=\frac{du}{-\sin(x)}$

So $\int(1-\cos^2(x))\sin(x)\,dx$

$= \int(1-u^2)\sin(x)\frac{du}{-\sin(x)}$

$= \int(u^2-1)\,du$

$= \frac{1}{3}u^3-u$

$= \frac{1}{3}\cos^3(x)-\cos(x)$

b $\int \cos^3(2x)\,dx$

$= \int(1-\sin^2(2x))\cos(2x)\,dx$

Let $u=\sin(2x)$

$\frac{du}{dx}=2\cos(2x)$ or $dx=\frac{du}{2\cos(2x)}$

So $\int(1-\sin^2(2x))\cos(2x)\,dx$

$= \int(1-u^2)\cos(2x)\frac{du}{2\cos(2x)}$

$= \frac{1}{2}\int(1-u^2)\,du$

$= \frac{1}{2}\left(u-\frac{1}{3}u^3\right)$

$= \frac{1}{2}\sin(2x)-\frac{1}{6}\sin^3(2x)$

c $\int 6\sin^3(4x)\,dx$

$= 6\int(1-\cos^2(4x))\sin(4x)\,dx$

Let $u = \cos(4x)$

$\frac{du}{dx} = -4\sin(4x)$ or $dx = \frac{du}{-4\sin(4x)}$

So $6\int(1-\cos^2(4x))\sin(4x)\,dx$

$= 6\int(1-u^2)\sin(4x)\frac{du}{-4\sin(4x)}$

$= \frac{-6}{4}\int(1-u^2)\,du$

$= \frac{3}{2}\int(u^2-1)\,du$

$= \frac{3}{2}\left(\frac{1}{3}u^3-u\right)$

$= \frac{1}{2}\cos^3(4x) - \frac{3}{2}\cos(4x)$

d $\int 4\cos^3(3x)\,dx$

$= 4\int(1-\sin^2(3x))\cos(3x)\,dx$

Let $u = \sin(3x)$

$\frac{du}{dx} = 3\cos(3x)$ or $dx = \frac{du}{3\cos(3x)}$

So $4\int(1-\sin^2(3x))\cos(3x)\,dx$

$= 4\int(1-u^2)\cos(3x)\frac{du}{3\cos(3x)}$

$= \frac{4}{3}\int(1-u^2)\,du$

$= \frac{4}{3}\left(u-\frac{1}{3}u^3\right)$

$= \frac{4}{3}\sin(3x) - \frac{4}{9}\sin^3(3x)$

e $\int \sin^3(7x)\,dx$

$= \int(1-\cos^2(7x))\sin(7x)\,dx$

Let $u = \cos(7x)$

$\frac{du}{dx} = -7\sin(7x)$ or $dx = \frac{du}{-7\sin(7x)}$

So $\int(1-\cos^2(7x))\sin(7x)\,dx$

$= \int(1-u^2)\sin(7x)\frac{du}{-7\sin(7x)}$

$= \frac{1}{7}\int(u^2-1)\,du$

$= \frac{1}{7}\left(\frac{1}{3}u^3-u\right)$

$= \frac{1}{21}\cos^3(7x) - \frac{1}{7}\cos(7x)$

f $\int \cos^3(6x)\,dx$

$= \int(1-\sin^2(6x))\cos(6x)\,dx$

Let $u = \sin(6x)$

$\frac{du}{dx} = 6\cos(6x)$ or $dx = \frac{du}{6\cos(6x)}$

So $\int(1-\sin^2(6x))\cos(6x)\,dx$

$= \int(1-u^2)\cos(6x)\frac{du}{6\cos(6x)}$

$= \frac{1}{6}\int(1-u^2)\,du$

$= \frac{1}{6}\left(u-\frac{1}{3}u^3\right)$

$= \frac{1}{6}\sin(6x) - \frac{1}{18}\sin^3(6x)$

g $\int 3\sin^3\left(\frac{x}{2}\right)dx$

$= 3\int\left(1-\cos^2\left(\frac{x}{2}\right)\right)\sin\left(\frac{x}{2}\right)dx$

Let $u = \cos\left(\frac{x}{2}\right)$

$\frac{du}{dx} = \frac{-1}{2}\sin\left(\frac{x}{2}\right)$ or $dx = \frac{-2\,du}{\sin\left(\frac{x}{2}\right)}$

So $3\int\left(1-\cos^2\left(\frac{x}{2}\right)\right)\sin\left(\frac{x}{2}\right)dx$

$= 3\int(1-u^2)\sin\left(\frac{x}{2}\right)\times\frac{-2\,du}{\sin\left(\frac{x}{2}\right)}$

$= 6\int(u^2-1)\,du$

$= 6\left(\frac{1}{3}u^3-u\right)$

$= 2\cos^3\left(\frac{x}{2}\right) - 6\cos\left(\frac{x}{2}\right)$

h $\int 2\cos^3\left(\frac{x}{3}\right)$

$= 2\int\left(1-\sin^2\left(\frac{x}{3}\right)\right)\cos\left(\frac{x}{3}\right)dx$

Let $u = \sin\left(\frac{x}{3}\right)$

$\frac{du}{dx} = \frac{1}{3}\cos\left(\frac{x}{3}\right)$ or $dx = \frac{3\,du}{\cos\left(\frac{x}{3}\right)}$

So $2\int\left(1-\sin^2\left(\frac{x}{3}\right)\right)\cos\left(\frac{x}{3}\right)dx$

$= 2\int(1-u^2)\cos\left(\frac{x}{3}\right)\times\frac{3\,du}{\cos\left(\frac{x}{3}\right)}$

$= 6\int(1-u^2)\,du$

$= 6\left(u-\frac{1}{3}u^3\right)$

$= 6\sin\left(\frac{x}{3}\right) - 2\sin^3\left(\frac{x}{3}\right)$

i $\int \sin^3\left(\frac{3x}{2}\right)dx$

$= \int\left(1-\cos^2\left(\frac{3x}{2}\right)\right)\sin\left(\frac{3x}{2}\right)dx$

Let $u = \cos\left(\frac{3x}{2}\right)$

$\frac{du}{dx} = \frac{-3}{2}\sin\left(\frac{3x}{2}\right)$ or $dx = \frac{-2\,du}{3\sin\left(\frac{3x}{2}\right)}$

So $\int\left(1-\cos^2\left(\frac{3x}{2}\right)\right)\sin\left(\frac{3x}{2}\right)dx$

$= \int (1-u^2)\sin\left(\frac{3x}{2}\right) \times \frac{-2\,du}{3\sin\left(\frac{3x}{2}\right)}$

$= \frac{2}{3}\int (u^2-1)\,du$

$= \frac{2}{3}\left(\frac{1}{3}u^3 - u\right)$

$= \frac{2}{9}\cos^3\left(\frac{3x}{2}\right) - \frac{2}{3}\cos\left(\frac{3x}{2}\right)$

j $\int \cos^3\left(\frac{5x}{2}\right)dx$

$= \int \left(1-\sin^2\left(\frac{5x}{2}\right)\right)\cos\left(\frac{5x}{2}\right)dx$

Let $u = \sin\left(\frac{5x}{2}\right)$

$\frac{du}{dx} = \frac{5}{2}\cos\left(\frac{5x}{2}\right)$ or $dx = \frac{2\,du}{5\cos\left(\frac{5x}{2}\right)}$

So $\int \left(1-\sin^2\left(\frac{5x}{2}\right)\right)\cos\left(\frac{5x}{2}\right)dx$

$= \int (1-u^2)\cos\left(\frac{5x}{2}\right) \times \frac{2\,du}{5\cos\left(\frac{5x}{2}\right)}$

$= \frac{2}{5}\int (1-u^2)\,du$

$= \frac{2}{5}\left(u - \frac{1}{3}u^3\right)$

$= \frac{2}{5}\sin\left(\frac{5x}{2}\right) - \frac{2}{15}\sin^3\left(\frac{5x}{2}\right)$

k $\int \sin^3\left(\frac{3x}{4}\right)dx$

$= \int \left(1-\cos^2\left(\frac{3x}{4}\right)\right)\sin\left(\frac{3x}{4}\right)dx$

Let $u = \cos\left(\frac{3x}{4}\right)$

$\frac{du}{dx} = \frac{-3}{4}\sin\left(\frac{3x}{4}\right)$ or $dx = \frac{-4\,du}{3\sin\left(\frac{3x}{4}\right)}$

So $\int \left(1-\cos^2\left(\frac{3x}{4}\right)\right)\sin\left(\frac{3x}{4}\right)dx$

$= \int (1-u^2)\sin\left(\frac{3x}{4}\right) \times \frac{-4\,du}{3\sin\left(\frac{3x}{4}\right)}$

$= \frac{4}{3}\int (u^2-1)\,du$

$= \frac{4}{3}\left(\frac{1}{3}u^3 - u\right)$

$= \frac{4}{9}\cos^3\left(\frac{3x}{4}\right) - \frac{4}{3}\cos\left(\frac{3x}{4}\right)$

l $\int \cos^3\left(\frac{4x}{3}\right)dx$

$= \int \left(1-\sin^2\frac{4x}{3}\right)\cos\left(\frac{4x}{3}\right)dx$

Let $u = \sin\left(\frac{4x}{3}\right)$

$\frac{du}{dx} = \frac{4}{3}\cos\left(\frac{4x}{3}\right)$ or $dx = \frac{3\,du}{4\cos(4x)}$

So $\int \left(1-\sin^2\left(\frac{4x}{3}\right)\right)\cos\left(\frac{4x}{3}\right) \times \frac{3\,du}{4\cos(4x)}$

$= \frac{3}{4}\int (1-u^2)\,du$

$= \frac{3}{4}\left(u - \frac{1}{3}u^3\right)$

$= \frac{3}{4}\sin\left(\frac{4x}{3}\right) - \frac{1}{4}\sin^3\left(\frac{4x}{3}\right)$

5 a $\int \sin(x)\cos(2x)\,dx$

$= \int \sin(x)(2\cos^2(x)-1)\,dx$

$= \int (2\cos^2(x)\sin(x) - \sin(x))\,dx$

$= 2\int \cos^2(x)\sin(x)\,dx - \int \sin(x)\,dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

So $2\int \cos^2(x)\sin(x)\,dx - \int \sin(x)\,dx$

$= 2\int u^2\sin(x)\frac{du}{-\sin(x)} - \int \sin(x)\,dx$

$= -2\int u^2\,du - \int \sin(x)\,dx$

$= -2\left(\frac{1}{3}u^3\right) + \cos(x) + c$

$= \frac{-2}{3}\cos^3(x) + \cos(x) + c$

$= \cos(x) - \frac{2}{3}\cos^3(x) + c$

b $\int \cos(2x)\cos(4x)\,dx$

$= \int \cos(2x)(1-2\sin^2(2x))\,dx$

$= \int (\cos(2x) - 2\sin^2(2x)\cos(2x))\,dx$

$= \int \cos(2x)\,dx - 2\int \sin^2(2x)\cos(2x)\,dx$

Let $u = \sin(2x)$

$\frac{du}{dx} = 2\cos(2x)$ or $dx = \frac{du}{2\cos(2x)}$

So $\int \cos(2x)\,dx - 2\int \sin^2(2x)\cos(2x)\,dx$

$= \int \cos(2x)\,dx - 2\int u^2\cos(2x)\frac{du}{2\cos(2x)}$

$= \int \cos(2x)\,dx - \int u^2\,du$

$= \frac{1}{2}\sin(2x) - \frac{1}{3}u^3 + c$

$= \frac{1}{2}\sin(2x) - \frac{1}{3}\sin^3(2x) + c$

c $\int \sin(3x)\cos(6x)\,dx$

$= \int \sin(3x)(2\cos^2(3x)-1)\,dx$

$= \int (2\cos^2(3x)\sin(3x) - \sin(3x))\,dx$

$= 2\int \cos^2(3x)\sin(3x)\,dx - \int \sin(3x)\,dx$

Let $u = \cos(3x)$

$\frac{du}{dx} = -3\sin(3x)$ or $dx = \frac{du}{-3\sin(3x)}$

So $2\int \cos^2(3x)\sin(3x)\,dx - \int \sin(3x)\,dx$

$= 2\int u^2 \sin(3x)\dfrac{du}{-3\sin(3x)} - \int \sin(3x)\,dx$

$= \dfrac{-2}{3}\int u^2\,du - \int \sin(3x)\,dx$

$= \dfrac{-2}{3}\left(\dfrac{1}{3}u^3\right) + \dfrac{1}{3}\cos(3x) + c$

$= \dfrac{1}{3}\cos(3x) - \dfrac{2}{9}u^3 + c$

$= \dfrac{1}{3}\cos(3x) - \dfrac{2}{9}\cos^3(3x) + c$

d $\int \cos(4x)\cos(8x)\,dx$

$= \int \cos(4x)(1 - 2\sin^2(4x))\,dx$

$= \int (\cos(4x) - 2\sin^2(4x)\cos(4x))\,dx$

$= \int \cos(4x)\,dx - 2\int \sin^2(4x)\cos(4x)\,dx$

Let $u = \sin(4x)$

$\dfrac{du}{dx} = 4\cos(4x)$ or $dx = \dfrac{du}{4\cos(4x)}$

So $\int \cos(4x)\,dx - 2\int \sin^2(4x)\cos(4x)\,dx$

$= \int \cos(4x)\,dx - 2\int u^2\cos(4x)\dfrac{du}{4\cos(4x)}$

$= \int \cos(4x)\,dx - \dfrac{1}{2}\int u^2\,du$

$= \dfrac{1}{4}\sin(4x) - \dfrac{1}{2}\left(\dfrac{1}{3}u^3\right) + c$

$= \dfrac{1}{4}\sin(4x) - \dfrac{1}{6}\sin^3(4x) + c$

e $\int \sin\left(\dfrac{x}{2}\right)\cos(x)\,dx$

$= \int \sin\left(\dfrac{x}{2}\right)\left(2\cos^2\left(\dfrac{x}{2}\right) - 1\right)dx$

$= \int \left(2\cos^2\left(\dfrac{x}{2}\right)\sin\left(\dfrac{x}{2}\right) - \sin\left(\dfrac{x}{2}\right)\right)dx$

$= 2\int \cos^2\left(\dfrac{x}{2}\right)\sin\left(\dfrac{x}{2}\right)dx - \int \sin\left(\dfrac{x}{2}\right)dx$

Let $u = \cos\left(\dfrac{x}{2}\right)$

$\dfrac{du}{dx} = \dfrac{-1}{2}\sin\left(\dfrac{x}{2}\right)$ or $dx = \dfrac{-2\,du}{\sin\left(\dfrac{x}{2}\right)}$

So $2\int \cos^2\left(\dfrac{x}{2}\right)\sin\left(\dfrac{x}{2}\right)dx - \int \sin\left(\dfrac{x}{2}\right)dx$

$= 2\int u^2 \sin\left(\dfrac{x}{2}\right) \times \dfrac{-2\,du}{\sin\left(\dfrac{x}{2}\right)} - \int \sin\left(\dfrac{x}{2}\right)dx$

$= -4\int u^2\,du - \int \sin\left(\dfrac{x}{2}\right)dx$

$= -4\left(\dfrac{1}{3}u^3\right) + 2\cos\left(\dfrac{x}{2}\right) + c$

$= 2\cos\left(\dfrac{x}{2}\right) - \dfrac{4}{3}u^3 + c$

$= 2\cos\left(\dfrac{x}{2}\right) - \dfrac{4}{3}\cos^3\left(\dfrac{x}{2}\right) + c$

f $\int \cos\left(\dfrac{x}{3}\right)\cos\left(\dfrac{2x}{3}\right)dx$

$= \int \cos\left(\dfrac{x}{3}\right)\left(1 - 2\sin^2\left(\dfrac{x}{3}\right)\right)dx$

$= \int \left(\cos\left(\dfrac{x}{3}\right) - 2\sin^2\left(\dfrac{x}{3}\right)\cos\left(\dfrac{x}{3}\right)\right)dx$

$= \int \cos\left(\dfrac{x}{3}\right)dx - 2\int \sin^2\left(\dfrac{x}{3}\right)\cos\left(\dfrac{x}{3}\right)dx$

Let $u = \sin\left(\dfrac{x}{3}\right)$

$\dfrac{du}{dx} = \dfrac{1}{3}\cos\left(\dfrac{x}{3}\right)$ or $dx = \dfrac{3\,du}{\cos\left(\dfrac{x}{3}\right)}$

So $\int \cos\left(\dfrac{x}{3}\right)dx - 2\int \sin^2\left(\dfrac{x}{3}\right)\cos\left(\dfrac{x}{3}\right)dx$

$= \int \cos\left(\dfrac{x}{3}\right)dx - 2\int u^2\cos\left(\dfrac{x}{3}\right) \times \dfrac{3\,du}{\cos\left(\dfrac{x}{3}\right)}$

$= \int \cos\left(\dfrac{x}{3}\right)dx - 6\int u^2\,du$

$= 3\sin\left(\dfrac{x}{3}\right) - 6\left(\dfrac{1}{3}u^3\right) + c$

$= 3\sin\left(\dfrac{x}{3}\right) - 2\sin^3\left(\dfrac{x}{3}\right) + c$

6 a $\int \sin(x)\cos^4(x)\,dx$

Let $u = \cos(x)$

$\dfrac{du}{dx} = -\sin(x)$ or $dx = \dfrac{du}{-\sin(x)}$

So $\int \sin(x)\cos^4(x)\,dx$

$= \int \sin(x)u^4\dfrac{du}{-\sin(x)}$

$= -\int u^4\,du$

$= \dfrac{-1}{5}u^5 + c$

$= \dfrac{-1}{5}\cos^5(x) + c$

b $\int \sin(2x)\cos^3(2x)\,dx$

Let $u = \cos(2x)$

$\dfrac{du}{dx} = -2\sin(2x)$ or $dx = \dfrac{du}{-2\sin(2x)}$

So $\int \sin(2x)\cos^3(2x)\,dx$

$= \int \sin(2x)u^3\dfrac{du}{-2\sin(2x)}$

$= \dfrac{-1}{2}\int u^3\,du$

$= \dfrac{-1}{2}\left(\dfrac{1}{4}u^4\right) + c$

$= \dfrac{-1}{8}\cos^4(2x) + c$

c $\int \sin\left(\dfrac{x}{2}\right)\cos^5\left(\dfrac{x}{2}\right)dx$

Let $u = \cos\left(\dfrac{x}{2}\right)$

$\frac{du}{dx} = \frac{-1}{2}\sin\left(\frac{x}{2}\right)$ or $dx = \frac{-2\,du}{\sin\left(\frac{x}{2}\right)}$

So $\int \sin\left(\frac{x}{2}\right)\cos^5\left(\frac{x}{2}\right)dx$

$= \int \sin\left(\frac{x}{2}\right)u^5 \times \frac{-2\,du}{\sin\left(\frac{x}{2}\right)}$

$= -2\int u^5\,du$

$= -2\left(\frac{1}{6}u^6\right) + c$

$= \frac{-1}{3}\cos^6\left(\frac{x}{2}\right) + c$

d $\int \cos(3x)\sin^4(3x)\,dx$

Let $u = \sin(3x)$

$\frac{du}{dx} = 3\cos(3x)$ or $dx = \frac{du}{3\cos(3x)}$

So $\int \cos(3x)\sin^4(3x)\,dx$

$= \int \cos(3x)u^4 \frac{du}{3\cos(3x)}$

$= \frac{1}{3}\int u^4\,du$

$= \frac{1}{3}\left(\frac{1}{5}u^5\right) + c$

$= \frac{1}{15}\sin^5(3x) + c$

e $\int \cos\left(\frac{x}{5}\right)\sin^6\left(\frac{x}{5}\right)dx$

Let $u = \sin\left(\frac{x}{5}\right)$

$\frac{du}{dx} = \frac{1}{5}\cos\left(\frac{x}{5}\right)$ or $dx = \frac{5\,du}{\cos\left(\frac{x}{5}\right)}$

So $\int \cos\left(\frac{x}{5}\right)\sin^6\left(\frac{x}{5}\right)dx$

$= \int \cos\left(\frac{x}{5}\right)u^6 \times \frac{5\,du}{\cos\left(\frac{x}{5}\right)}$

$= 5\int u^6\,du$

$= 5\left(\frac{1}{7}u^7\right) + c$

$= \frac{5}{7}\sin^7\left(\frac{x}{5}\right) + c$

f $\int \cos\left(\frac{2x}{3}\right)\sin^7\left(\frac{2x}{3}\right)dx$

Let $u = \sin\left(\frac{2x}{3}\right)$

$\frac{du}{dx} = \frac{2}{3}\cos\left(\frac{2x}{3}\right)$ or $dx = \frac{3\,du}{2\cos\left(\frac{2x}{3}\right)}$

So $\int \cos\left(\frac{2x}{3}\right)\sin^7\left(\frac{2x}{3}\right)dx$

$= \int \cos\left(\frac{2x}{3}\right)u^7 \times \frac{3\,du}{2\cos\left(\frac{2x}{3}\right)}$

$= \frac{3}{2}\int u^7\,du$

$= \frac{3}{2}\left(\frac{1}{8}u^8\right) + c$

$= \frac{3}{16}\sin^8\left(\frac{2x}{3}\right) + c$

7 a $\int \cos^2(x)\sin^3(x)\,dx$

$= \int \cos^2(x)\sin^2(x)\sin(x)\,dx$

$= \int \cos^2(x)(1-\cos^2(x))\sin(x)\,dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

So $\int \cos^2(x)(1-\cos^2(x))\sin(x)\,dx$

$= \int u^2(1-u^2)\sin(x)\frac{du}{-\sin(x)}$

$= \int u^2(u^2-1)\,du$

$= \int (u^4-u^2)\,du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + c$

$= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + c$

b $\int \sin^2(x)\cos^3(x)\,dx$

$= \int \sin^2(x)\cos^2(x)\cos(x)\,dx$

$= \int \sin^2(x)(1-\sin^2(x))\cos(x)\,dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

So $\int \sin^2(x)(1-\sin^2(x))\cos(x)\,dx$

$= \int u^2(1-u^2)\cos(x)\frac{du}{\cos(x)}$

$= \int (u^2-u^4)\,du$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + c$

$= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + c$

c $\int \cos^2(2x)\sin^3(2x)\,dx$

$= \int \cos^2(2x)\sin^2(2x)\sin(2x)\,dx$

$= \int \cos^2(2x)(1-\cos^2(2x))\sin(2x)\,dx$

Let $u = \cos(2x)$

$\frac{du}{dx} = -2\sin(2x)$ or $dx = \frac{du}{-2\sin(2x)}$

So $\int \cos^2(2x)(1-\cos^2(2x))\sin(2x)\,dx$

$= \int u^2(1-u^2)\sin(2x)\frac{du}{-2\sin(2x)}$

$= \frac{1}{2}\int u^2(u^2-1)\,du$

$= \frac{1}{2}\int (u^4-u^2)\,du$

$= \frac{1}{2}\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) + c$

$= \frac{1}{10}\cos^5(2x) - \frac{1}{6}\cos^3(2x) + c$

d $\int \sin^2(3x)\cos^3(3x)\,dx$

$= \int \sin^2(3x)\cos^2(3x)\cos(3x)\,dx$

$= \int \sin^2(3x)(1-\sin^2(3x))\cos(3x)\,dx$

Let $u = \sin(3x)$

$\frac{du}{dx} = 3\cos(3x)$ or $dx = \frac{du}{3\cos(3x)}$

So $\int \sin^2(3x)(1-\sin^2(3x))\cos(3x)\,dx$

$= \int u^2(1-u^2)\cos(3x)\frac{du}{3\cos(3x)}$

$= \frac{1}{3}\int (u^2 - u^4)\,du$

$= \frac{1}{3}\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + c$

$= \frac{1}{9}\sin^3(3x) - \frac{1}{15}\sin^5(3x) + c$

e $\int \cos^2\left(\frac{x}{2}\right)\sin^3\left(\frac{x}{2}\right)dx$

$= \int \cos^2\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx$

$= \int \cos^2\left(\frac{x}{2}\right)\left(1-\cos^2\left(\frac{x}{2}\right)\right)\sin\left(\frac{x}{2}\right)dx$

Let $u = \cos\left(\frac{x}{2}\right)$

$\frac{du}{dx} = \frac{-1}{2}\sin\left(\frac{x}{2}\right)$ or $dx = \frac{-2\,du}{\sin\left(\frac{x}{2}\right)}$

So $\int \cos^2\left(\frac{x}{2}\right)\left(1-\cos^2\left(\frac{x}{2}\right)\right)\sin\left(\frac{x}{2}\right)dx$

$= \int u^2(1-u^2)\sin\left(\frac{x}{2}\right)\times\frac{-2\,du}{\sin\left(\frac{x}{2}\right)}$

$= 2\int u^2(u^2-1)\,du$

$= 2\int (u^4 - u^2)\,du$

$= 2\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) + c$

$= \frac{2}{5}\cos^5\left(\frac{x}{2}\right) - \frac{2}{5}\cos^3\left(\frac{x}{2}\right) + c$

f $\int \sin^2\left(\frac{3x}{2}\right)\cos^3\left(\frac{3x}{2}\right)dx$

$= \int \sin^2\left(\frac{3x}{2}\right)\cos^2\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)dx$

$= \int \sin^2\left(\frac{3x}{2}\right)\left(1-\sin^2\left(\frac{3x}{2}\right)\right)\cos\left(\frac{3x}{2}\right)dx$

Let $u = \sin\left(\frac{3x}{2}\right)$

$\frac{du}{dx} = \frac{3}{2}\cos\left(\frac{3x}{2}\right)$ or $dx = \frac{2\,du}{3\cos\left(\frac{3x}{2}\right)}$

So $\int \sin^2\left(\frac{3x}{2}\right)\left(1-\sin^2\left(\frac{3x}{2}\right)\right)\cos\left(\frac{3x}{2}\right)dx$

$= \int u^2(1-u^2)\cos\left(\frac{3x}{2}\right)\times\frac{2\,du}{3\cos\left(\frac{3x}{2}\right)}$

$= \frac{2}{3}\int u^2(1-u^2)\,du$

$= \frac{2}{3}\int (u^2 - u^4)\,du$

$= \frac{2}{3}\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + c$

$= \frac{2}{9}\sin^3\left(\frac{3x}{2}\right) - \frac{2}{15}\sin^5\left(\frac{3x}{2}\right) + c$

g $\int 4\cos^2\left(\frac{x}{3}\right)\sin^3\left(\frac{x}{3}\right)dx$

$= 4\int \cos^2\left(\frac{x}{3}\right)\sin^2\left(\frac{x}{3}\right)\sin\left(\frac{x}{3}\right)dx$

$= 4\int \cos^2\left(\frac{x}{3}\right)\left(1-\cos^2\left(\frac{x}{3}\right)\right)\sin\left(\frac{x}{3}\right)dx$

Let $u = \cos\left(\frac{x}{3}\right)$

$\frac{du}{dx} = \frac{-1}{3}\sin\left(\frac{x}{3}\right)$ or $dx = \frac{-3\,du}{\sin\left(\frac{x}{3}\right)}$

So $4\int \cos^2\left(\frac{x}{3}\right)\left(1-\cos^2\left(\frac{x}{3}\right)\right)\sin\left(\frac{x}{3}\right)dx$

$= 4\int u^2(1-u^2)\sin\left(\frac{x}{3}\right)\times\frac{-3\,du}{\sin\left(\frac{x}{3}\right)}$

$= 12\int u^2(u^2-1)\,du$

$= 12\int (u^4 - u^2)\,du$

$= 12\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) + c$

$= \frac{12}{5}\cos^5\left(\frac{x}{3}\right) - 4\cos^3\left(\frac{x}{3}\right) + c$

h $\int -6\sin^2\left(\frac{5x}{4}\right)\cos^3\left(\frac{5x}{4}\right)dx$

$= -6\int \sin^2\left(\frac{5x}{4}\right)\cos^2\left(\frac{5x}{4}\right)\cos\left(\frac{5x}{4}\right)dx$

$= -6\int \sin^2\left(\frac{5x}{4}\right)\left(1-\sin^2\left(\frac{5x}{4}\right)\right)\cos\left(\frac{5x}{4}\right)dx$

Let $u = \sin\left(\frac{5x}{4}\right)$

$\frac{du}{dx} = \frac{5}{4}\cos\left(\frac{5x}{4}\right)$ or $dx = \frac{4\,du}{5\cos\left(\frac{5x}{4}\right)}$

So $-6\int \sin^2\left(\frac{5x}{4}\right)\left(1-\sin^2\left(\frac{5x}{4}\right)\right)\cos\left(\frac{5x}{4}\right)dx$

$= -6\int u^2(1-u^2)\cos\left(\frac{5x}{4}\right)\times\frac{4\,du}{5\cos\left(\frac{5x}{4}\right)}$

$= \frac{24}{5}\int u^2(u^2-1)\,du$

$= \frac{24}{5}\int(u^4 - u^2)\,du$

$= \frac{24}{5}\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) + c$

$= \frac{24}{25}\sin^5\left(\frac{5x}{4}\right) - \frac{24}{15}\sin^3\left(\frac{5x}{4}\right) + c$

$= \frac{24}{25}\sin^5\left(\frac{5x}{4}\right) - \frac{8}{5}\sin^3\left(\frac{5x}{4}\right) + c$

i $\int \sin^3(x)\cos^4(x)\,dx$

$= \int \sin(x)\sin^2(x)\cos^4(x)\,dx$

$= \int \sin(x)(1-\cos^2(x))\cos^4(x)\,dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

So $\int \sin(x)(1-\cos^2(x))\cos^4(x)\,dx$

$= \int \sin(x)(1-u^2)u^4 \frac{du}{-\sin(x)}$

$= \int (u^2-1)u^4\,du$

$= \int (u^6 - u^4)\,du$

$= \frac{1}{7}u^7 - \frac{1}{5}u^5 + c$

$= \frac{1}{7}\cos^7(x) - \frac{1}{5}\cos^5(x) + c$

j $\int \cos^3(2x)\sin^4(2x)\,dx$

$= \int \cos(2x)\cos^2(2x)\sin^4(2x)\,dx$

$= \int \cos(2x)(1-\sin^2(2x))\sin^4(2x)\,dx$

Let $u = \sin(2x)$

$\frac{du}{dx} = 2\cos(2x)$ or $dx = \frac{du}{2\cos(2x)}$

So $\int \cos(2x)(1-\sin^2(2x))\sin^4(2x)\,dx$

$= \int \cos(2x)(1-u^2)u^4 \frac{du}{2\cos(2x)}$

$= \frac{1}{2}\int (1-u^2)u^4\,du$

$= \frac{1}{2}\int (u^4 - u^6)\,du$

$= \frac{1}{2}\left(\frac{1}{5}u^5 - \frac{1}{7}u^7\right) + c$

$= \frac{1}{10}\sin^5(2x) - \frac{1}{14}\sin^7(2x) + c$

k $\int 2\sin^3(2x)\cos^5(2x)\,dx$

$= 2\int \sin(2x)\sin^2(2x)\cos^5(2x)\,dx$

$= 2\int \sin(2x)(1-\cos^2(2x))\cos^5(2x)\,dx$

Let $u = \cos(2x)$

$\frac{du}{dx} = -2\sin(2x)$ or $dx = \frac{du}{-2\sin(2x)}$

So $2\int \sin(2x)(1-\cos^2(2x))\cos^5(2x)\,dx$

$= 2\int \sin(2x)(1-u^2)u^5 \frac{du}{-2\sin(2x)}$

$= \int (u^2-1)u^5\,du$

$= \int (u^7 - u^5)\,du$

$= \frac{1}{8}u^8 - \frac{1}{6}u^6 + c$

$= \frac{1}{8}\cos^8(2x) - \frac{1}{6}\cos^6(2x) + c$

l $\int -2\cos^3(3x)\sin^6(3x)\,dx$

$= -2\int \cos(3x)\cos^2(3x)\sin^6(3x)\,dx$

$= -2\int \cos(3x)(1-\sin^2(3x))\sin^6(3x)\,dx$

Let $u = \sin(3x)$

$\frac{du}{dx} = 3\cos(3x)$ or $dx = \frac{du}{3\cos(3x)}$

So $-2\int \cos(3x)(1-\sin^2(3x))\sin^6(3x)\,dx$

$= -2\int \cos(3x)(1-u^2)u^6 \frac{du}{3\cos(3x)}$

$= \frac{2}{3}\int (u^2-1)u^6\,du$

$= \frac{2}{3}\int (u^8 - u^6)\,du$

$= \frac{2}{3}\left(\frac{1}{9}u^9 - \frac{1}{7}u^7\right) + c$

$= \frac{2}{27}\sin^9(3x) - \frac{2}{21}\sin^7(3x) + c$

m $\int 4\sin^3\left(\frac{x}{2}\right)\cos^6\left(\frac{x}{2}\right)dx$

$= 4\int \sin\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)\cos^6\left(\frac{x}{2}\right)dx$

$= 4\int \sin\left(\frac{x}{2}\right)\left(1-\cos^2\left(\frac{x}{2}\right)\right)\cos^6\left(\frac{x}{2}\right)dx$

Let $u = \cos\left(\frac{x}{2}\right)$

$\frac{du}{dx} = \frac{-1}{2}\sin\left(\frac{x}{2}\right)$ or $dx = \frac{-2\,du}{\sin\left(\frac{x}{2}\right)}$

So $4\int \sin\left(\frac{x}{2}\right)\left(1-\cos^2\left(\frac{x}{2}\right)\right)\cos^6\left(\frac{x}{2}\right)dx$

$= 4\int \sin\left(\frac{x}{2}\right)(1-u^2)\,u^6 \times \frac{-2\,du}{\sin\left(\frac{x}{2}\right)}$

$= 8\int (u^2-1)u^6\,du$

$= 8\int (u^8 - u^6)\,du$

$= 8\left(\frac{1}{9}u^9 - \frac{1}{7}u^7\right) + c$

$= \frac{8}{9}\cos^9\left(\frac{x}{2}\right) - \frac{8}{7}\cos^7\left(\frac{x}{2}\right) + c$

n $\int \cos^3\left(\frac{3x}{2}\right)\sin^7\left(\frac{3x}{2}\right)dx$

$= \int \cos\left(\frac{3x}{2}\right)\cos^2\left(\frac{3x}{2}\right)\sin^7\left(\frac{3x}{2}\right)dx$

$= \int \cos\left(\frac{3x}{2}\right)\left(1-\sin^2\left(\frac{3x}{2}\right)\right)\sin^7\left(\frac{3x}{2}\right)dx$

Let $u = \sin\left(\frac{3x}{2}\right)$

$\frac{du}{dx} = \frac{3}{2}\cos\left(\frac{3x}{2}\right)$ or $dx = \frac{2\,du}{3\cos\left(\frac{3x}{2}\right)}$

So $\int \cos\left(\frac{3x}{2}\right)\left(1-\sin^2\left(\frac{3x}{2}\right)\right)\sin^7\left(\frac{3x}{2}\right)dx$

$= \int \cos\left(\frac{3x}{2}\right)(1-u^2)u^7 \times \frac{2\,du}{3\cos\left(\frac{3x}{2}\right)}$

$= \frac{2}{3}\int (1-u^2)u^7\,du$

$= \frac{2}{3}\int (u^7 - u^9)\,du$

$= \frac{2}{3}\left(\frac{1}{8}u^8 - \frac{1}{10}u^{10}\right) + c$

$= \frac{1}{12}\sin^8\left(\frac{3x}{2}\right) - \frac{1}{15}\sin^{10}\left(\frac{3x}{2}\right) + c$

8 a $\int (1+\tan^2(2x))\,dx$

$= \int \sec^2(2x)\,dx$

$= \frac{1}{2}\tan(2x)$

b $\int \left(1+\tan^2\left(\frac{x}{3}\right)\right)dx$

$= \int \sec^2\left(\frac{x}{3}\right)dx$

$= 3\tan\left(\frac{x}{3}\right)$

c $\int \tan^2(x)\sec^2(x)\,dx$

Let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$ or $dx = \frac{du}{\sec^2(x)}$

So $\int \tan^2(x)\sec^2(x)\,dx$

$= \int u^2 \sec^2(x)\frac{du}{\sec^2(x)}$

$= \int u^2\,du$

$= \frac{1}{3}u^3$

$= \frac{1}{3}\tan^3(x)$

d $\int \tan^3(x)\sec^2(x)\,dx$

Let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$ or $dx = \frac{du}{\sec^2(x)}$

So $\int \tan^3(x)\sec^2(x)\,dx$

$= \int u^3 \sec^2(x)\frac{du}{\sec^2(x)}$

$= \int u^3\,du$

$= \frac{1}{4}u^4$

$= \frac{1}{4}\tan^4(x)$

e $\int 4\tan^5(2x)\sec^2(2x)\,dx$

Let $u = \tan(2x)$

$\frac{du}{dx} = 2\sec^2(2x)$ or $dx = \frac{du}{2\sec^2(2x)}$

So $\int 4\tan^5(2x)\sec^2(2x)\,dx$

$= \int 4u^5 \sec^2(2x)\frac{du}{2\sec^2(2x)}$

$= 2\int u^5\,du$

$= 2\left(\frac{1}{6}u^6\right)$

$= \frac{1}{3}\tan^6(2x)$

f $\int 8\tan^4\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx$

Let $u = \tan\left(\frac{x}{2}\right)$

$\frac{du}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$ or $dx = \frac{2\,du}{\sec^2\left(\frac{x}{2}\right)}$

So $\int 8\tan^4\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx$

$= \int 8u^4\sec^2\left(\frac{x}{2}\right) \times \frac{2\,du}{\sec^2\left(\frac{x}{2}\right)}$

$= 16\int u^4\,du$

$= 16\left(\frac{1}{5}u^5\right)$

$= \frac{16}{5}\tan^5\left(\frac{x}{2}\right)$

g $\int \tan^2(x)\sec^4(x)\,dx$

$= \int \tan^2(x)\sec^2(x)\sec^2(x)\,dx$

$= \int \tan^2(x)(1+\tan^2(x))\sec^2(x)\,dx$

Let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$ or $dx = \frac{du}{\sec^2(x)}$

So $\int \tan^2(x)(1+\tan^2(x))\sec^2(x)\,dx$

$= \int u^2(1+u^2)\sec^2(x)\frac{du}{\sec^2(x)}$

$= \int (u^2+u^4)\,du$

$= \frac{1}{3}u^3 + \frac{1}{5}u^5$

$= \frac{1}{3}\tan^3(x) + \frac{1}{5}\tan^5(x)$

h $\int 6\tan^2(2x)\sec^4(2x)\,dx$

$= 6\int \tan^2(2x)\sec^2(2x)\sec^2(2x)\,dx$

$= 6\int \tan^2(2x)(1+\tan^2(2x))\sec^2(2x)\,dx$

Let $u = \tan(2x)$

$\frac{du}{dx} = 2\sec^2(2x)$ or $dx = \frac{du}{2\sec^2(2x)}$

So $6\int \tan^2(2x)(1+\tan^2(2x))\sec^2(2x)\,dx$

$= 6\int u^2(1+u^2)\sec^2(2x)\frac{du}{2\sec^2(2x)}$

$= 3\int (u^2+u^4)\,du$

$= 3\left(\frac{1}{3}u^3 + \frac{1}{5}u^5\right)$

$= \tan^3(2x) + \frac{3}{5}\tan^5(2x)$

i $\int 2\tan^2\left(\frac{x}{2}\right)\sec^4\left(\frac{x}{2}\right)dx$

$= 2\int \tan^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx$

$= 2\int \tan^2\left(\frac{x}{2}\right)\left(1+\tan^2\left(\frac{x}{2}\right)\right)\sec^2\left(\frac{x}{2}\right)dx$

Let $u = \tan\left(\frac{x}{2}\right)$

$\frac{du}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$ or $dx = \frac{2\,du}{\sec^2\left(\frac{x}{2}\right)}$

So $2\int \tan^2\left(\frac{x}{2}\right)\left(1+\tan^2\left(\frac{x}{2}\right)\right)\sec^2\left(\frac{x}{2}\right)dx$

$= 2\int u^2(1+u^2)\sec^2\left(\frac{x}{2}\right)\times\frac{2\,du}{\sec^2\left(\frac{x}{2}\right)}$

$= 4\int (u^2+u^4)\,du$

$= 4\left(\frac{1}{3}u^3 + \frac{1}{5}u^5\right)$

$= \frac{4}{3}\tan^3\left(\frac{x}{2}\right) + \left(\frac{4}{5}\right)\tan^5\left(\frac{x}{2}\right)$

j $\int 3\tan^3(3x)\sec^4(3x)\,dx$

$= 3\int \tan^3(3x)\sec^2(3x)\sec^2(3x)\,dx$

$= 3\int \tan^3(3x)(1+\tan^2(3x))\sec^2(3x)\,dx$

Let $u = \tan(3x)$

$\frac{du}{dx} = 3\sec^2(3x)$ or $dx = \frac{du}{3\sec^2(3x)}$

So $3\int \tan^3(3x)(1+\tan^2(3x))\sec^2(3x)\,dx$

$= 3\int u^3(1+u^2)\sec^2(3x)\frac{du}{3\sec^2(3x)}$

$= \int (u^3+u^5)\,du$

$= \frac{1}{4}u^4 + \frac{1}{6}u^6$

$= \frac{1}{4}\tan^4(3x) + \frac{1}{6}\tan^6(3x)$

k $\int \tan^4\left(\frac{x}{5}\right)\sec^4\left(\frac{x}{5}\right)dx$

$= \int \tan^4\left(\frac{x}{5}\right)\sec^2\left(\frac{x}{5}\right)\sec^2\left(\frac{x}{5}\right)dx$

$= \int \tan^4\left(\frac{x}{5}\right)\left(1+\tan^2\left(\frac{x}{5}\right)\right)\sec^2\left(\frac{x}{5}\right)dx$

Let $u = \tan\left(\frac{x}{5}\right)$

$\frac{du}{dx} = \frac{1}{5}\sec^2\left(\frac{x}{5}\right)$ or $dx = \frac{5\,du}{\sec^2\left(\frac{x}{5}\right)}$

So $\int \tan^4\left(\frac{x}{5}\right)\left(1+\tan^2\left(\frac{x}{5}\right)\right)\sec^2\left(\frac{x}{5}\right)dx$

$= \int u^4(1+u^2)\sec^2\left(\frac{x}{5}\right)\times\frac{5\,du}{\sec^2\left(\frac{x}{5}\right)}$

$= 5\int (u^4-u^6)\,du$

$= 5\left(\frac{1}{5}u^5 - \frac{1}{7}u^7\right)$

$= \tan^5\left(\frac{x}{5}\right) - \frac{5}{7}\tan^7\left(\frac{x}{5}\right)$

l $\int 12\tan^5(6x)\sec^6(6x)\,dx$

$= 12\int \tan^5(6x)(\sec^2(6x))^2\sec^2(x)\,dx$

$= 12\int \tan^5(6x)(1+\tan^2(6x))^2\sec^2(x)\,dx$

Let $u = \tan(6x)$

$\frac{du}{dx} = 6\sec^2(6x)$ or $dx = \frac{du}{6\sec^2(6x)}$

So $12\int \tan^5(6x)(1+\tan^2(6x))^2\sec^2(x)\,dx$

$= 12\int u^5(1+u^2)^2\sec^2(x)\frac{du}{6\sec^2(6x)}$

$= 2\int u^5(1+u^2)^2\,du$

$= 2\int u^5(1+2u^2+u^4)\,du$

$= 2\int (u^5+2u^7+u^9)\,du$

$= 2\left(\frac{1}{6}u^6 + \frac{2}{8}u^8 + \frac{1}{10}u^{10}\right)$

$= \frac{1}{3}u^6 + \frac{1}{2}u^8 + \frac{1}{5}u^{10}$

$= \frac{1}{3}\tan^6(6x) + \frac{1}{2}\tan^8(6x) + \frac{1}{5}\tan^{10}(6x)$

9 a $\int \sin(x)\cos^n(x)\,dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

So $\int \sin(x)\cos^n(x)\,dx$

$= \int \sin(x)u^n\frac{du}{-\sin(x)}$

$= -\int u^n\,du$

$= \frac{-1}{n+1}u^{n+1} + c$

$= \frac{-\cos^{n+1}(x)}{n+1} + c$

b $\int \cos(x)\sin^n(x)\,dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

So $\int \cos(x)\sin^n(x)\,dx$

$= \int \cos(x)u^n\frac{du}{\cos(x)}$

$= \int u^n\,du$

$= \frac{1}{n+1}u^{n+1} + c$

$= \frac{\sin^{n+1}(x)}{n+1} + c$

c $\int \sec^2(x)\tan^n(x)\,dx$

Let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$ or $dx = \frac{du}{\sec^2(x)}$

So $\int \sec^2(x)\tan^n(x)\,dx$

$= \int \sec^2(x)u^n \frac{du}{\sec^2(x)}$

$= \int u^n\,du$

$= \frac{1}{n+1}u^{n+1} + c$

$= \frac{\tan^{n+1}(x)}{n+1} + c$

d $\int \sin^3(x)\cos^n(x)\,dx$

$= \int \sin(x)\sin^2(x)\cos^n(x)\,dx$

$= \int \sin(x)(1-\cos^2(x))\cos^n(x)\,dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

So $\int \sin(x)(1-\cos^2(x))\cos^n(x)\,dx$

$= \int \sin(x)(1-u^2)u^n \frac{du}{-\sin(x)}$

$= \int (u^2-1)u^n\,du$

$= \int (u^{n+2}-u^n)\,du$

$= \frac{1}{n+3}u^{n+3} - \frac{1}{n+1}u^{n+1} + c$

$= \frac{\cos^{n+3}(x)}{n+3} - \frac{\cos^{n+1}(x)}{n+1} + c$

e $\int \cos^3(x)\sin^n(x)\,dx$

$= \int \cos(x)\cos^2(x)\sin^n(x)\,dx$

$= \int \cos(x)(1-\sin^2(x))\sin^n(x)\,dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

So $\int \cos(x)(1-\sin^2(x))\sin^n(x)\,dx$

$= \int \cos(x)(1-u^2)u^n \frac{du}{\cos(x)}$

$= \int (u^n - u^{n+2})\,du$

$= \frac{1}{n+1}u^{n+1} - \frac{1}{n+3}u^{n+3} + c$

$= \frac{\sin^{n+1}(x)}{n+1} - \frac{\sin^{n+3}(x)}{n+3} + c$

10 a If $f'(x) = 6\sin(x)\cos^2(x)$ and $f\left(\frac{\pi}{3}\right) = 0$, find $f(x)$

$f'(x) = 6\sin(x)\cos^2(x)$

$f(x) = \int 6\sin(x)\cos^2(x)\,dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

So $f(x) = \int 6\sin(x)u^2 \frac{du}{-\sin(x)}$

$= -6\int u^2\,du$

$= -6\left(\frac{1}{3}u^3\right) + c$

$f(x) = -2\cos^3(x) + c$

$f\left(\frac{\pi}{3}\right) = -2\cos^3\left(\frac{\pi}{3}\right) + c = 0$

$-2\left(\frac{1}{2}\right)^3 + c = 0$

$\frac{-2}{8} + c = 0$

$c = \frac{1}{4}$

So $f(x) = -2\cos^3(x) + \frac{1}{4}$

11 If $f'(x) = 4\sin^2(2x)\cos^2(2x)$ and $f\left(\frac{\pi}{4}\right) = \pi$, find $f(x)$

$f'(x) = 4\sin^2(2x)\cos^2(2x)$

$f(x) = \int 4\sin^2(2x)\cos^2(2x)\,dx$

$= \int (2\sin(2x)\cos(2x))^2\,dx$

$= \int \sin^2(4x)\,dx$

$= \int \frac{1}{2}(1-\cos(8x))\,dx$

$= \frac{1}{2}\left(x - \frac{1}{8}\sin(8x)\right) + c$

$f(x) = \frac{x}{2} - \frac{1}{16}\sin(8x) + c$

$f\left(\frac{\pi}{4}\right) = \frac{\pi}{8} - \frac{1}{16}\sin(2\pi) + c = \pi$

$\frac{\pi}{8} + c = \pi$

$c = \frac{7\pi}{8}$

Therefore $f(x) = \frac{x}{2} - \frac{1}{16}\sin(8x) + \frac{7\pi}{8}$

12 Find $g(x)$ if $g'(x) = \sin^3\left(\frac{x}{2}\right)\cos^4\left(\frac{x}{2}\right)$ and $g(0) = \frac{-4}{35}$

$g'(x) = \sin^3\left(\frac{x}{2}\right)\cos^4\left(\frac{x}{2}\right)$

$g(x) = \int \sin^3\left(\frac{x}{2}\right)\cos^4\left(\frac{x}{2}\right)dx$

$= \int \sin\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)\cos^4\left(\frac{x}{2}\right)dx$

$= \int \sin\left(\frac{x}{2}\right)\left(1-\cos^2\left(\frac{x}{2}\right)\right)\cos^4\left(\frac{x}{2}\right)dx$

Let $u = \cos\left(\frac{x}{2}\right)$

$\frac{du}{dx} = \frac{-1}{2}\sin\frac{x}{2}$ or $dx = \frac{-2\,du}{\sin\left(\frac{x}{2}\right)}$

So $g(x) = \int \sin\left(\frac{x}{2}\right)(1-u^2)u^4 \times \frac{-2\,du}{\sin\left(\frac{x}{2}\right)}$

$= 2\int (u^2-1)u^4\,du$

$= 2\int (u^6-u^4)\,du$

$= 2\left(\frac{1}{7}u^7 - \frac{1}{5}u^5\right) + c$

$g(x) = \frac{2}{7}\cos^7\left(\frac{x}{2}\right) - \frac{2}{5}\cos^5\left(\frac{x}{2}\right) + c$

$g(0) = \frac{2}{7}\cos^7(0) - \frac{2}{5}\cos^5(0) + c = \frac{-4}{35}$

$\frac{2}{7} - \frac{2}{5} + c = \frac{-4}{35}$

$\frac{10-14}{35} + c = \frac{-4}{35}$

$\frac{-4}{35} + c = \frac{-4}{35}$

$c = 0$

Therefore $g(x) = \frac{2}{7}\cos^7\left(\frac{x}{2}\right) - \frac{2}{5}\cos^5\left(\frac{x}{2}\right)$

Exercise 4E — Antidifferentiation using partial fractions

1 a $ax + b(x - 1) = 3x - 2$

Let $x = 0$, $-b = -2$

$b = 2$

Let $x = 1$, $a = 3 - 2$

$a = 1$

b $a(x + 2) + b(x - 3) = x - 8$

Let $x = -2$, $-5b = -2 - 8$

$-5b = -10$

$b = 2$

Let $x = 3$, $5a = 3 - 8$

$5a = -5$

$a = -1$

c $a(x - 4) + b = 3x - 2$

Let $x = 4$, $b = 3 \times 4 - 2$

$b = 10$

Let $x = 0$, $-4a + 10 = -2$

$-4a = -12$

$a = 3$

d $a(3x + 1) + b(x - 2) = 5x + 4$

Let $x = \frac{-1}{3}$, $b\left(\frac{-1}{3} - 2\right) = 5\left(\frac{-1}{3}\right) + 4$

$\frac{-7}{3}b = \frac{-5+12}{3}$

$-7b = 7$

$b = -1$

Let $x = 2$, $a(2 \times 3 + 1) = 5 \times 2 + 4$

$7a = 14$

$a = 2$

e $a(2 - 3x) + b(x + 5) = 9x + 11$

Let $x = -5$, $a(2 + 15) = -45 + 11$

$17a = -34$

$a = -2$

Let $x = \frac{2}{3}$, $b\left(\frac{2}{3} + 5\right) = 9 \times \left(\frac{2}{3}\right) + 11$

$\frac{17}{3}b = 6 + 11$

$\frac{17}{3}b = 17$

$b = 3$

f $a(x + 2) + bx = 2x - 10$

Let $x = 0$, $2a = -10$

$a = -5$

Let $x = -2$, $-2b = -4 - 10$

$-2b = -14$

$b = 7$

g $a + b(x + 2) + c(x + 2)(x + 3) = x^2 + 4x - 2$

Let $x = -2$, $a = 4 - 8 - 2$

$a = -6$

Let $x = -3$, $-6 - b = 9 - 12 - 2$

$-6 - b = -5$

$-b = 1$

$b = -1$

Let $x = 0$, $-6 - 2 + 6c = -2$

$-8 + 6c = -2$

$6c = 6$

$c = 1$

h $a(x + 2)(x - 3) + bx(x - 3) + cx(x + 2) = 3x^2 - x + 6$

Let $x = -2$, $b \times (-2) \times (-5) = 3 \times 4 + 2 + 6$

$10b = 12 + 8$

$10b = 20$

$b = 2$

Let $x = 0$, $-6a = 6$

$a = -1$

Let $x = 3$, $15c = 3 \times 9 - 3 + 6$

$15c = 27 + 3$

$15c = 30$

$c = 2$

2 a $\frac{1}{(x+1)(x+2)} = \frac{a}{x+1} + \frac{b}{x+2}$

$= \frac{a(x+2) + b(x+1)}{(x+1)(x+2)}$

So $1 = a(x + 2) + b(x + 1)$

Let $x = -2$, $1 = -b$

$b = -1$

Let $x = -1$, $1 = a$

$a = 1$

Therefore $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$

b $\frac{12}{(x-2)(x+2)} = \frac{a}{x-2} + \frac{b}{x+2}$

$= \frac{a(x+2) + b(x-2)}{(x-2)(x+2)}$

So $12 = a(x + 2) + b(x - 2)$

Let $x = -2$, $12 = -4b$

$b = -3$

Let $x = 2$, $12 = 4a$

$a = 3$

So $\frac{12}{(x-2)(x+2)} = \frac{3}{x-2} - \frac{3}{x+2}$

c $\frac{6x}{(x+3)(x-1)} = \frac{a}{x+3} + \frac{b}{x-1}$

$= \frac{a(x-1) + b(x+3)}{(x+3)(x-1)}$

So $6x = a(x - 1) + b(x + 3)$

Let $x = -3$, $-18 = -4a$

$a = \frac{9}{2}$

Let $x = 1$, $6 = 4b$

$b = \frac{3}{2}$

Therefore $\frac{6x}{(x+3)(x-1)} = \frac{9}{2(x+3)} + \frac{3}{2(x-1)}$

d $\frac{3x}{(x-2)(x+1)} = \frac{a}{x-2} + \frac{b}{x+1}$

$= \frac{a(x+1) + b(x-2)}{(x-2)(x+1)}$

So $3x = a(x+1) + b(x-2)$
Let $x = -1$, $-3 = -3b$
$b = 1$
Let $x = 2$, $6 = 3a$
$a = 2$
Therefore $\dfrac{3x}{(x-2)(x+1)} = \dfrac{2}{x-2} + \dfrac{1}{x+1}$

e $\dfrac{x+3}{(x+2)(x+3)} = \dfrac{\cancel{x+3}}{(x+2)\cancel{(x+3)}}$ $(x \neq -3)$
$= \dfrac{1}{x+2}$

f $\dfrac{x+20}{(x-4)(x+4)} = \dfrac{a}{x-4} + \dfrac{b}{x+4}$
$= \dfrac{a(x+4)+b(x-4)}{(x-4)(x+4)}$
So $x + 20 = a(x+4) + b(x-4)$
Let $x = -4$, $16 = -8b$
$b = -2$
Let $x = 4$, $24 = 8a$
$a = 3$
Therefore $\dfrac{x+20}{(x-4)(x+4)} = \dfrac{3}{x-4} - \dfrac{2}{x+4}$

g $\dfrac{4x+5}{(x+2)^2} = \dfrac{a}{x+2} + \dfrac{b}{(x+2)^2}$
$= \dfrac{a(x+2)+b}{(x+2)^2}$
So $4x + 5 = a(x+2) + b$
Let $x = -2$, $-8 + 5 = b$
$b = -3$
Let $x = 0$, $5 = 2a - 3$
$8 = 2a$
$a = 4$
Therefore $\dfrac{4x+5}{(x+2)^2} = \dfrac{4}{x+2} - \dfrac{3}{(x+2)^2}$

h $\dfrac{5x-26}{(x-5)^2} = \dfrac{a}{x-5} + \dfrac{b}{(x-5)^2}$
$= \dfrac{a(x-5)+b}{(x-5)^2}$
So $5x - 26 = a(x-5) + b$
Let $x = 5$, $25 - 26 = b$
$b = -1$
Let $x = 0$, $-26 = -5a - 1$
$-25 = -5a$
$a = 5$
Therefore $\dfrac{5x-26}{(x-5)^2} = \dfrac{5}{x-5} - \dfrac{1}{(x-5)^2}$

i $\dfrac{x+4}{x(x-2)} = \dfrac{a}{x} + \dfrac{b}{x-2}$
$= \dfrac{a(x-2)+bx}{x(x-2)}$
So $x + 4 = a(x-2) + bx$
Let $x = 0$, $4 = -2a$
$a = -2$
Let $x = 2$, $6 = 2b$
$b = 3$
Therefore $\dfrac{x+4}{x(x-2)} = \dfrac{-2}{x} + \dfrac{3}{x-2}$

j $\dfrac{7x-4}{(x-2)(x+3)} = \dfrac{a}{x-2} + \dfrac{b}{x+3}$
$= \dfrac{a(x+3)+b(x-2)}{(x-2)(x+3)}$
So $7x - 4 = a(x+3) + b(x-2)$
Let $x = -3$, $-21 - 4 = -5b$
$-25 = -5b$
$b = 5$
Let $x = 2$, $14 - 4 = 5a$
$10 = 5a$
$a = 2$
Therefore $\dfrac{7x-4}{(x-2)(x+3)} = \dfrac{2}{x-2} + \dfrac{5}{x+3}$

k $\dfrac{8x-10}{(2x+1)(x-3)} = \dfrac{a}{2x+1} + \dfrac{b}{x-3}$
$= \dfrac{a(x-3)+b(2x+1)}{(2x+1)(x-3)}$
So $8x - 10 = a(x-3) + b(2x+1)$
Let $x = \dfrac{-1}{2}$, $-4 - 10 = \dfrac{-7}{2}a$
$-14 = \dfrac{-7}{2}a$
$a = 4$
Let $x = 3$, $24 - 10 = 7b$
$14 = 7b$
$b = 2$
Therefore $\dfrac{8x-10}{(2x+1)(x-3)} = \dfrac{4}{2x+1} + \dfrac{2}{x-3}$

l $\dfrac{9x-11}{(3x-2)(x+1)} = \dfrac{a}{3x-2} + \dfrac{b}{x+1}$
$= \dfrac{a(x+1)+b(3x-2)}{(3x-2)(x+1)}$
So $9x - 11 = a(x+1) + b(3x-2)$
Let $x = -1$, $-9 - 11 = -5b$
$-20 = -5b$
$b = 4$
Let $x = \dfrac{2}{3}$, $6 - 11 = \dfrac{5}{3}a$
$-5 = \dfrac{5}{3}a$
$a = -3$
Therefore $\dfrac{9x-11}{(3x-2)(x+1)} = \dfrac{-3}{3x-2} + \dfrac{4}{x+1}$

m $\dfrac{11-3x}{(2-x)(x+3)} = \dfrac{a}{2-x} + \dfrac{b}{x+3}$
$= \dfrac{a(x+3)+b(2-x)}{(2-x)(x+3)}$
So $11 - 3x = a(x+3) + b(2-x)$
Let $x = -3$, $11 + 9 = 5b$
$20 = 5b$
$b = 4$
Let $x = 2$, $11 - 6 = 5a$
$5 = 5a$
$a = 1$
Therefore $\dfrac{11-3x}{(2-x)(x+3)} = \dfrac{1}{2-x} + \dfrac{4}{x+3}$

n $\dfrac{12-2x}{(1-x)(3-x)} = \dfrac{a}{1-x} + \dfrac{b}{3-x}$
$= \dfrac{a(3-x)+b(1-x)}{(1-x)(3-x)}$

So $12 - 2x = a(3 - x) + b(1 - x)$

Let $x = 1$, $12 - 2 = 2a$

$10 = 2a$

$a = 5$

Let $x = 3$, $12 - 6 = -2b$

$6 = -2b$

$b = -3$

Therefore $\dfrac{12 - 2x}{(1 - x)(3 - x)} = \dfrac{5}{1 - x} - \dfrac{3}{3 - x}$

3 a $\int \frac{1}{(x+1)(x+2)}\,dx$

$= \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right)dx$

$= \log_e(x + 1) - \log_e(x + 2) + c \quad (x > -1)$

$= \log_e\left(\frac{x+1}{x+2}\right) + c$

b $\int \frac{12}{(x-2)(x+2)}\,dx$

$= \int \left(\frac{3}{x-2} - \frac{3}{x+2}\right)dx$

$= 3\log_e(x - 2) - 3\log_e(x + 2) + c \quad (x > 2)$

$= \log_e[(x - 2)^3] - \log_e[(x + 2)^3] + c$

$= \log_e\left[\left(\frac{x-2}{x+2}\right)^3\right] + c$

$= 3\log_e\left(\frac{x-2}{x+2}\right) + c$

c $\int \frac{6x}{(x+3)(x-1)}\,dx = \int \left(\frac{9}{2(x+3)} + \frac{3}{2(x-1)}\right)dx$

$= \frac{9}{2}\log_e(x+3) + \frac{3}{2}\log_e(x-1) + c \quad (x > 1)$

$= \log_e\left[(x+3)^{\frac{9}{2}}\right] + \log_e\left[(x-1)^{\frac{3}{2}}\right] + c$

$= \log_e\left[(x+3)^{\frac{9}{2}}(x-1)^{\frac{3}{2}}\right] + c$

$= \frac{1}{2}\log_e\left[(x+3)^9(x-1)^3\right] + c$

d $\int \frac{3x}{(x-2)(x+1)}\,dx$

$= \int \left(\frac{2}{x-2} + \frac{1}{x+1}\right)dx$

$= 2\log_e(x - 2) + \log_e(x + 1) + c \quad (x > 2)$

e $\int \frac{x+3}{(x+2)(x+3)}\,dx = \int \frac{1}{x+2}\,dx \quad (x \neq -3)$

$= \log_e(x + 2) + c \quad (x > -2)$

f $\int \frac{x+20}{(x-4)(x+4)}\,dx$

$= \int \left(\frac{3}{x-4} - \frac{2}{x+4}\right)dx$

$= 3\log_e(x - 4) - 2\log_e(x + 4) + c \quad (x > 4)$

g $\int \frac{4x+5}{(x+2)^2}\,dx$

$= \int \left(\frac{4}{x+2} - \frac{3}{(x+2)^2}\right)dx$

$= \int \left(\frac{4}{x+2} - 3(x+2)^{-2}\right)dx$

$= 4\log_e(x + 2) + 3(x + 2)^{-1} + c \quad (x > -2)$

$= 4\log_e(x + 2) + \frac{3}{x+2} + c$

h $\int \frac{5x-26}{(x-5)^2}\,dx$

$= \int \left(\frac{5}{x-5} - \frac{1}{(x-5)^2}\right)dx$

$= \int \left(\frac{5}{x-5} - (x-5)^{-2}\right)dx$

$= 5\log_e(x - 5) + (x - 5)^{-1} + c \quad (x > 5)$

$= 5\log_e(x - 5) + \frac{1}{x-5} + c$

i $\int \frac{x+4}{x(x-2)}\,dx$

$= \int \left(\frac{-2}{x} + \frac{3}{x-2}\right)dx$

$= -2\log_e(x) + 3\log_e(x - 2) + c \quad (x > 2)$

$= 3\log_e(x - 2) - 2\log_e(x)$

j $\int \frac{7x-4}{(x-2)(x+3)}\,dx$

$= \int \left(\frac{2}{x-2} + \frac{5}{x+3}\right)dx$

$= 2\log_e(x - 2) + 5\log_e(x + 3) + c \quad (x > 2)$

k $\int \frac{8x-10}{(2x+1)(x-3)}\,dx$

$= \int \left(\frac{4}{2x+1} + \frac{2}{x-3}\right)dx$

$= \frac{4}{2}\log_e(2x + 1) + 2\log_e(x - 3) + c \quad (x > 3)$

$= 2\log_e(2x + 1) + 2\log_e(x - 3) + c$

l $\int \frac{9x-11}{(3x-2)(x+1)}\,dx$

$= \int \left(\frac{-3}{3x-2} + \frac{4}{x+1}\right)dx$

$= \frac{-3}{3}\log_e(3x - 2) + 4\log_e(x + 1) + c \quad \left(x > \frac{2}{3}\right)$

$= 4\log_e(x + 1) - \log_e(3x - 2) + c$

m $\int \frac{11-3x}{(2-x)(x+3)}\,dx$

$= \int \left(\frac{1}{2-x} + \frac{4}{x+3}\right)dx$

$= -\log_e(2 - x) + 4\log_e(x + 3) + c$

$(x < 2 \cap x > -3 \Rightarrow -3 < x < 2)$

$= 4\log_e(x + 3) - \log_e(2 - x) + c$

n $\int \frac{12-2x}{(1-x)(3-x)}\,dx$

$= \int \left(\frac{5}{1-x} - \frac{3}{3-x}\right)dx$

$= -5\log_e(1 - x) + 3\log_e(3 - x) + c \quad (x < 1)$

$= 3\log_e(3 - x) - 5\log_e(1 - x) + c$

4 a $\dfrac{5x + 10}{24 - 2x - x^2} = \dfrac{5x + 10}{(x + 6)(4 - x)}$

$= \dfrac{a}{x + 6} + \dfrac{b}{4 - x}$

$= \dfrac{a(4-x)+b(x+6)}{(x+6)(4-x)}$

So $5x+10 = a(4-x)+b(x+6)$

Let $x=-6$, $-30+10 = 10a$

$-20 = 10a$

$a = -2$

Let $x = 4$, $20+10 = 10b$

$30 = 10b$

$b = 3$

$\therefore$ **D**

b $\displaystyle\int \frac{5x+10}{24-2x-x^2}dx$

$= \displaystyle\int \left(\frac{-2}{x+6}+\frac{3}{4-x}\right)dx$

$= -2\log_e(x+6) - 3\log_e(4-x) + c$

$\therefore$ **B**

5 $\dfrac{-10}{x^2+x-6} = \dfrac{-10}{(x+3)(x-2)}$

$= \dfrac{a}{x+3}+\dfrac{b}{x-2}$

$= \dfrac{a(x-2)+b(x+3)}{(x+3)(x-2)}$

So $-10 = a(x-2)+b(x+3)$

Let $x=-3$, $-10 = -5a$

$a = 2$

Let $x = 2$, $-10 = 5b$

$b = -2$

So $\displaystyle\int \frac{-10}{x^2+x-6}dx$

$= \displaystyle\int \left(\frac{2}{x+3}-\frac{2}{x-2}\right)dx$

$= 2\log_e(x+3) - 2\log_e(x-2) + c$

$= 2[\log_e(x+3) - \log_e(x-2)] + c$

$= 2\log_e\left(\dfrac{x+3}{x-2}\right)+c$

$\therefore$ **C**

6 a $\displaystyle\int \frac{3x+10}{x^2+2x}dx$

$\dfrac{3x+10}{x^2+2x} = \dfrac{3x+10}{x(x+2)}$

$= \dfrac{a}{x}+\dfrac{b}{x+2}$

$= \dfrac{a(x+2)+bx}{x(x+2)}$

So $3x+10 = a(x+2)+bx$

Let $x=-2$, $-6+10 = -2b$

$4 = -2b$

$b = -2$

Let $x = 0$, $10 = 2a$

$a = 5$

Therefore $\displaystyle\int \frac{3x+10}{x^2+2x}$

$= \displaystyle\int \left(\frac{5}{x}-\frac{2}{x+2}\right)dx$

$= 5\log_e(x) - 2\log_e(x+2) + c \quad (x > 0)$

b $\displaystyle\int \frac{5x-4}{x^2-x-2}$

$\dfrac{5x-4}{x^2-x-2} = \dfrac{5x-4}{(x-2)(x+1)}$

$= \dfrac{a}{x-2}+\dfrac{b}{x+1}$

$= \dfrac{a(x+1)+b(x-2)}{(x-2)(x+1)}$

So $5x-4 = a(x+1)+b(x-2)$

Let $x=-1$, $-5-4 = -3b$

$-9 = -3b$

$b = 3$

Let $x = 2$, $10-4 = 3a$

$6 = 3a$

$a = 2$

Therefore $\displaystyle\int \frac{5x-4}{x^2-x+2}dx$

$= \displaystyle\int \left(\frac{2}{x-2}+\frac{3}{x+1}\right)dx$

$= 2\log_e(x-2) + 3\log_e(x+1) + c \quad (x > 2)$

c $\displaystyle\int \frac{x+3}{x^2+3x+2}dx$

$\dfrac{x+3}{x^2+3x+2} = \dfrac{x+3}{(x+1)(x+2)}$

$= \dfrac{a}{x+1}+\dfrac{b}{x+2}$

$= \dfrac{a(x+2)+b(x+1)}{(x+1)(x+2)}$

So $x+3 = a(x+2)+b(x+1)$

Let $x=-2$, $1 = -b$

$b = -1$

Let $x=-1$, $2 = a$

$a = 2$

Therefore $\displaystyle\int \frac{x+3}{x^2+3x+2}dx$

$= \displaystyle\int \left(\frac{2}{x+1}-\frac{1}{x+2}\right)dx$

$= 2\log_e(x+1) - \log_e(x+2) + c \quad (x > -1)$

d $\displaystyle\int \frac{6x-1}{x^2-5x-6}dx$

$\dfrac{6x-1}{x^2-5x-6} = \dfrac{6x-1}{(x-6)(x+1)}$

$= \dfrac{a}{x-6}+\dfrac{b}{x+1}$

$= \dfrac{a(x+1)+b(x-6)}{(x-6)(x+1)}$

So $6x-1 = a(x+1)+b(x-6)$

Let $x=-1$, $-6-1 = -7b$

$-7 = -7b$

$b = 1$

Let $x = 6$, $36-1 = 7a$

$35 = 7a$

$a = 5$

Therefore $\displaystyle\int \frac{6x-1}{x^2-5x-6}dx$

$= \displaystyle\int \left(\frac{5}{x-6}+\frac{1}{x+1}\right)dx$

$= 5\log_e(x-6) + \log_e(x+1) + c \quad (x > 6)$

e $\displaystyle\int \frac{5x-7}{x^2-4x+3}$

$\dfrac{5x-7}{x^2-4x+3} = \dfrac{5x-7}{(x-3)(x-1)}$

$$= \frac{a}{x-3} + \frac{b}{x-1}$$
$$= \frac{a(x-1) + b(x-3)}{(x-3)(x-1)}$$

So $5x - 7 = a(x - 1) + b(x - 3)$

Let $x = 1$, $5 - 7 = -2b$
$-2 = -2b$
$b = 1$

Let $x = 3$, $15 - 7 = 2a$
$8 = 2a$
$a = 4$

Therefore $\int \frac{5x-7}{x^2 - 4x + 3} dx$

$$= \int \left(\frac{4}{x-3} + \frac{1}{x-1} \right) dx$$
$$= 4 \log_e(x-3) + \log_e(x-1) + c \quad (x > 3)$$

f $\int \frac{x+16}{x^2 + 7x + 6} dx$

$$\frac{x+16}{x^2 + 7x + 6} = \frac{x+16}{(x+1)(x+6)}$$
$$= \frac{a}{x+1} + \frac{b}{x+6}$$
$$= \frac{a(x+6) + b(x+1)}{(x+1)(x+6)}$$

So $x + 16 = a(x + 6) + b(x + 1)$

Let $x = -6$, $10 = -5b$
$b = -2$

Let $x = -1$, $15 = 5a$
$a = 3$

Therefore $\int \frac{x+16}{x^2 + 7x + 16} dx$

$$= \int \left(\frac{3}{x+1} - \frac{2}{x+6} \right) dx$$
$$= 3 \log_e(x+1) - 2 \log_e(x+6) + c \quad (x > -1)$$

g $\int \frac{7x+9}{x^2 - 9} dx$

$$\frac{7x+9}{x^2 - 9} = \frac{7x+9}{(x+3)(x-3)}$$
$$= \frac{a}{x+3} + \frac{b}{x-3}$$
$$= \frac{a(x-3) + b(x+3)}{(x+3)(x-3)}$$

So $7x + 9 = a(x - 3) + b(x + 3)$

Let $x = -3$, $-21 + 9 = -6a$
$-12 = -6a$
$a = 2$

Let $x = 3$, $21 + 9 = 6b$
$30 = 6b$
$b = 5$

Therefore $\int \frac{7x+9}{x^2 - 9} dx$

$$= \int \left(\frac{2}{x+3} + \frac{5}{x-3} \right) dx$$
$$= 2 \log_e(x+3) + 5 \log_e(x-3) + c \quad (x > 3)$$

h $\int \frac{7x+1}{x^2 - 1} dx$

$$\frac{7x+1}{x^2-1} = \frac{7x+1}{(x+1)(x-1)}$$
$$= \frac{a}{x+1} + \frac{b}{x-1}$$
$$= \frac{a(x-1)+b(x+1)}{(x+1)(x-1)}$$

So $7x + 1 = a(x - 1) + b(x + 1)$

Let $x = -1$, $-7 + 1 = -2a$
$-6 = -2a$
$a = 3$

Let $x = 1$, $7 + 1 = 2b$
$8 = 2b$
$b = 4$

Therefore $\int \frac{7x+1}{x^2-1} dx$

$$= \int \left(\frac{3}{x+1} + \frac{4}{x-1} \right) dx$$
$$= 3 \log_e(x+1) + 4 \log_e(x-1) + c \quad (x > 1)$$

i $\int \frac{5x}{2x^2 - 3x - 2} dx$

$$\frac{5x}{2x^2 - 3x - 2} = \frac{5x}{(2x+1)(x-2)}$$
$$= \frac{a}{2x+1} + \frac{b}{x-2}$$
$$= \frac{a(x-2)+b(2x+1)}{(x-2)(2x+1)}$$

So $5x = a(x - 2) + b(2x + 1)$

Let $x = -\frac{1}{2}$, $\frac{-5}{2} = \frac{-5}{2}a$
$a = 1$

Let $x = 2$, $10 = (4 + 1)b$
$10 = 5b$
$b = 2$

Therefore $\int \frac{5x}{2x^2 - 3x - 2} dx$

$$= \int \left(\frac{1}{2x+1} + \frac{2}{x-2} \right) dx$$
$$= \frac{1}{2} \log_e(2x+1) + 2 \log_e(x-2) + c \quad (x > 2)$$

j $\int \frac{16-2x}{3x^2 + 7x - 6} dx$

$$\frac{16-2x}{3x^2 + 7x - 6} = \frac{16-2x}{(3x-2)(x+3)}$$
$$= \frac{a}{3x-2} + \frac{b}{x+3}$$
$$= \frac{a(x+3)+b(3x-2)}{(3x-2)(x+3)}$$

So $16 - 2x = a(x + 3) + b(3x - 2)$

Let $x = -3$, $16 + 6 = (-9 - 2)b$
$22 = -11b$
$b = -2$

Let $x = \frac{2}{3}$, $16 - \frac{4}{3} = a\left(\frac{2}{3} + 3\right)$
$\frac{48-4}{3} = \frac{(2+9)}{3}a$
$44 = 11a$
$a = 4$

Therefore $\int \frac{16-2x}{3x^2 + 7x - 6} dx$

$$= \int \left(\frac{4}{3x-2} - \frac{2}{x+3} \right) dx$$

$= \frac{4}{3}\log_e(3x-2) - 2\log_e(x+3) + c$

$\left(x > \frac{2}{3}\right)$

k $\int \frac{x+4}{2x^2-5x+2}$

$\frac{x+4}{2x^2-5x+2} = \frac{x+4}{(x-2)(2x-1)}$

$= \frac{a}{x-2} + \frac{b}{2x-1}$

$= \frac{a(2x-1)+b(x-2)}{(x-2)(2x-1)}$

So $x + 4 = a(2x-1) + b(x-2)$

Let $x = \frac{1}{2}$, $\frac{1}{2} + 4 = \frac{-3}{2}b$

$\frac{9}{2} = \frac{-3}{2}b$

$b = -3$

Let $x = 2$, $6 = (4-1)a$

$6 = 3a$

$a = 2$

Therefore $\int \frac{x+4}{2x^2-5x+2}dx$

$= \int \left(\frac{2}{x-2} - \frac{3}{2x-1}\right)dx$

$= 2\log_e(x-2) - \frac{3}{2}\log_e(2x-1) + c$

$(x > 2)$

l $\int \frac{4}{4-x^2}dx$

$\frac{4}{4-x^2} = \frac{4}{(2+x)(2-x)}$

$= \frac{a}{2+x} + \frac{b}{2-x}$

$= \frac{a(2-x)+b(2+x)}{(2+x)(2-x)}$

So $4 = a(2-x) + b(2+x)$

Let $x = -2$, $4 = 4a$

$a = 1$

Let $x = 2$, $4 = 4b$

$b = 1$

Therefore $\int \frac{4}{4-x^2}dx$

$= \int \left(\frac{1}{2+x} + \frac{1}{2-x}\right)dx$

$= \log_e(2+x) - \log_e(2-x) + c$

$(x > -2 \cap x < 2 \Rightarrow -2 < x < 2)$

$= \log_e\left(\frac{2+x}{2-x}\right) + c$

m $\int \frac{3x-4}{16-x^2}dx$

$\frac{3x-4}{16-x^2} = \frac{3x-4}{(4-x)(4+x)}$

$= \frac{a}{4-x} + \frac{b}{4+x}$

$= \frac{a(4+x)+b(4-x)}{(4-x)(4+x)}$

So $3x - 4 = a(4+x) + b(4-x)$

Let $x = -4$, $-12 - 4 = 8b$

$-16 = 8b$

$b = -2$

Let $x = 4$, $12 - 4 = 8a$

$8 = 8a$

$a = 1$

Therefore $\int \frac{3x-4}{16-x^2}dx$

$= \int \left(\frac{1}{4-x} - \frac{2}{4+x}\right)dx$

$= -\log_e(4-x) - 2\log_e(4+x) + c$

$(-4 < x < 4)$

n $\int \frac{x+13}{5+4x-x^2}dx$

$\frac{x+13}{5+4x-x^2} = \frac{x+13}{(1+x)(5-x)}$

$= \frac{a}{1+x} + \frac{b}{5-x}$

$= \frac{a(5-x)+b(1+x)}{(1+x)(5-x)}$

So $x + 13 = a(5-x) + b(1+x)$

Let $x = -1$, $12 = 6a$

$a = 2$

Let $x = 5$, $18 = 6b$

$b = 3$

Therefore $\int \frac{x+13}{5+4x-x^2}dx$

$= \int \left(\frac{2}{1+x} + \frac{3}{5-x}\right)dx$

$= 2\log_e(1+x) - 3\log_e(5-x) + c$

$(x > -1 \cap x < 5 \Rightarrow -1 < x < 5)$

7 a $\int \frac{x-1}{x+5}dx$

$\frac{x-1}{x+5} = \frac{x+5-6}{x+5}$

$= \frac{x+5}{x+5} - \frac{6}{x+5}$

$= 1 - \frac{6}{x+5}$

Therefore $\int \frac{x-1}{x+5}dx$

$= \int \left(1 - \frac{6}{x+5}\right)dx$

$= x - 6\log_e(x+5) + c \quad (x > -5)$

b $\int \frac{x+3}{x-2}dx$

$\frac{x+3}{x-2} = \frac{x-2+5}{x-2}$

$= \frac{x-2}{x-2} + \frac{5}{x-2}$

$= 1 + \frac{5}{x-2}$

Therefore $\int \frac{x+3}{x-2}dx$

$= \int \left(1 + \frac{5}{x-2}\right)dx$

$= x + 5\log_e(x-2) + c \quad (x > 2)$

c $\int \frac{x^2-1}{x^2+3x}$

$$\frac{x^2-1}{x^2+3x} = \frac{x^2+3x-3x-1}{x^2+3x}$$
$$= \frac{x^2+3x}{x^2+3x}+\frac{-3x-1}{x^2+3x}$$
$$= 1+\frac{-3x-1}{x^2+3x}$$

Now $\frac{-3x-1}{x^2+3x} = \frac{-3x-1}{x(x+3)}$
$$= \frac{a}{x}+\frac{b}{x+3}$$
$$= \frac{a(x+3)+bx}{x(x+3)}$$

So $-3x-1 = a(x+3)+bx$

Let $x=-3$, $9-1=-3b$
$$8=-3b$$
$$b=\frac{-8}{3}$$

Let $x=0$, $-1=3a$
$$a=\frac{-1}{3}$$

So $\frac{-3x-1}{x^2+3x} = \frac{-1}{3x}-\frac{8}{3(x+3)}$

Therefore $\int \frac{x^2-1}{x^2+3x}\,dx$
$$= \int\left(1-\frac{1}{3x}-\frac{8}{3(x+3)}\right)dx$$
$$= x-\frac{1}{3}\log_e(x)-\frac{8}{3}\log_e(x+3)+c \quad (x>0)$$

d $\int \frac{x^2+2x+4}{x^2-4x}dx$

$$\frac{x^2+2x+4}{x^2-4x} = \frac{x^2-4x+6x+4}{x^2-4x}$$
$$= \frac{x^2-4x}{x^2-4x}+\frac{6x+4}{x^2-4x}$$
$$= 1+\frac{6x+4}{x^2-4x}$$

Now $\frac{6x+4}{x^2-4x} = \frac{6x+4}{(x-4)x}$
$$= \frac{a}{x-4}+\frac{b}{x}$$
$$= \frac{ax+b(x-4)}{(x-4)x}$$

So $6x+4 = ax+b(x-4)$

Let $x=0$, $4=-4b$
$$b=-1$$

Let $x=4$, $24+4=4a$
$$28=4a$$
$$a=7$$

So $\frac{6x+4}{x^2-4x} = \frac{7}{x-4}-\frac{1}{x}$

Therefore $\int \frac{6x+4}{x^2-4x}dx$
$$= \int\left(1+\frac{7}{x-4}-\frac{1}{x}\right)dx$$
$$= x+7\log_e(x-4)-\log_e(x)+c \quad (x>4)$$

e $\int \frac{x^2-x}{(x+3)(x+1)}dx$

$$\frac{x^2-x}{(x+3)(x+1)} = \frac{x^2-x}{x^2+4x+3}$$
$$= \frac{x^2+4x+3-5x-3}{x^2+4x+3}$$
$$= \frac{x^2+4x+3}{x^2+4x+3}+\frac{-5x-3}{x^2+4x+3}$$
$$= 1+\frac{-5x-3}{(x+1)(x+3)}$$

Now $\frac{-5x-3}{(x+3)(x+1)} = \frac{a}{x+1}+\frac{b}{x+3}$
$$= \frac{a(x+3)+b(x+1)}{(x+1)(x+3)}$$

So $-5x-3 = a(x+3)+b(x+1)$

Let $x=-3$, $15-3=-2b$
$$12=-2b$$
$$b=-6$$

Let $x=-1$, $5-3=2a$
$$2=2a$$
$$a=1$$

So $\frac{-5x-3}{(x+1)(x+3)} = \frac{1}{x+1}-\frac{6}{x+3}$

Therefore $\int \frac{x^2-x}{(x+3)(x+1)}dx$
$$= \int\left(1+\frac{1}{x+1}-\frac{6}{x+3}\right)dx$$
$$= x+\log_e(x+1)-6\log_e(x+3)+c \quad (x>-1)$$

f $\int \frac{x^2+x+4}{x^2-2x-3}dx$

$$\frac{x^2+x+4}{x^2-2x-3} = \frac{x^2-2x-3+3x+7}{x^2-2x-3}$$
$$= \frac{x^2-2x-3}{x^2-2x-3}+\frac{3x+7}{x^2-2x-3}$$
$$= 1+\frac{3x+7}{(x-3)(x+1)}$$

Now $\frac{3x+7}{(x-3)(x+1)} = \frac{a}{x-3}+\frac{b}{x+1}$
$$= \frac{a(x+1)+b(x-3)}{(x-3)(x+1)}$$

So $3x+7 = a(x+1)+b(x-3)$

Let $x=-1$, $-3+7=-4b$
$$4=-4b$$
$$b=-1$$

Let $x=3$, $9+7=4a$
$$16=4a$$
$$a=4$$

So $\frac{3x+7}{(x-3)(x+1)} = \frac{4}{x-3}-\frac{1}{x+1}$

Therefore $\int \frac{x^2+x+4}{x^2-2x-3}dx$
$$= \int\left(1+\frac{4}{x-3}-\frac{1}{x+1}\right)dx$$
$$= x+4\log_e(x-3)-\log_e(x+1)+c \quad (x>3)$$

g $\int \frac{x^2+3x-2}{x^2-4}\,dx$

$$\frac{x^2+3x-2}{x^2-4} = \frac{x^2-4+3x+2}{x^2-4}$$
$$= \frac{x^2-4}{x^2-4}+\frac{3x+2}{x^2-4}$$
$$= 1+\frac{3x+2}{(x+2)(x-2)}$$

Now $\frac{3x+2}{(x+2)(x-2)} = \frac{a}{x+2}+\frac{b}{x-2}$

$$= \frac{a(x-2)+b(x+2)}{(x+2)(x-2)}$$

So $3x+2 = a(x-2)+b(x+2)$

Let $x=-2$, $-6+2=-4a$

$-4=-4a$

$a=1$

Let $x=2$, $6+2=4b$

$8=4b$

$b=2$

So $\frac{3x+2}{(x+2)(x-2)} = \frac{1}{x+2}+\frac{2}{x-2}$

Therefore $\int \frac{x^2+3x-2}{x^2-4}\,dx$

$$= \int\left(1+\frac{1}{x+2}+\frac{2}{x-2}\right)dx$$
$$= x+\log_e(x+2)+2\log_e(x-2)+c$$
$(x>2)$

h $\int \frac{x^3+4x^2-x}{(x+2)(x+1)}\,dx$

$$\frac{x^3+4x^2-x}{(x+2)(x+1)} = \frac{x^3+4x^2-x}{x^2+3x+2}$$
$$= \frac{x^3+3x^2+2x+x^2-3x}{x^2+3x+2}$$
$$= \frac{x(x^2+3x+2)}{x^2+3x+2}+\frac{x^2-3x}{x^2+3x+2}$$
$$= x+\frac{x^2+3x+2-6x-2}{x^2+3x+2}$$
$$= x+\frac{x^2+3x+2}{x^2+3x+2}+\frac{-6x-2}{x^2+3x+2}$$
$$= x+1+\frac{-6x-2}{(x+2)(x+1)}$$

Now $\frac{-6x-2}{(x+2)(x+1)} = \frac{a}{x+2}+\frac{b}{x+1}$

$$= \frac{a(x+1)+b(x+2)}{(x+2)(x+1)}$$

So $-6x-2 = a(x+1)+b(x+2)$

Let $x=-2$, $12-2=-a$

$10=-a$

$a=-10$

Let $x=-1$, $6-2=b$

$b=4$

So $\frac{-6x-2}{(x+2)(x+1)} = \frac{-10}{x+2}+\frac{4}{x+1}$

Therefore $\int \frac{x^3+4x^2-x}{(x+2)(x+1)}\,dx$

$$= \int\left(x+1-\frac{10}{x+2}+\frac{4}{x+1}\right)dx$$
$$= \frac{1}{2}x^2+x-10\log_e(x+2)$$
$$+4\log_e(x+1)+c \quad (x>-1)$$

i $\int \frac{x^3+4x-13}{x^2-4x-5}\,dx$

$$\frac{x^3+4x-13}{x^2-4x-5} = \frac{x^3-4x^2-5x+4x^2+9x-13}{x^2-4x-5}$$
$$= \frac{x(x^2-4x-5)}{x^2-4x-5}+\frac{4x^2+9x-13}{x^2-4x-5}$$
$$= x+\frac{4x^2-16x-20+25x+7}{x^2-4x-5}$$
$$= x+\frac{4(x^2-4x-5)}{x^2-4x-5}+\frac{25x+7}{x^2-4x-5}$$
$$= x+4+\frac{25x+7}{(x-5)(x+1)}$$

Now $\frac{25x+7}{(x-5)(x+1)} = \frac{a}{x-5}+\frac{b}{x+1}$

$$= \frac{a(x+1)+b(x-5)}{(x-5)(x+1)}$$

So $25x+7 = a(x+1)+b(x-5)$

Let $x=-1$, $-25+7=-6b$

$-18=-6b$

$b=3$

Let $x=5$, $125+7=6a$

$132=6a$

$a=22$

So $\frac{25x+7}{(x-5)(x+1)} = \frac{22}{x-5}+\frac{3}{x+1}$

Therefore $\int \frac{x^3+4x-13}{x^2-4x-5}\,dx$

$$= \int\left(x+4+\frac{22}{x-5}+\frac{3}{x+1}\right)dx$$
$$= \frac{1}{2}x^2+4x+22\log_e(x-5)+$$
$$3\log_e(x+1)+c \quad (x>5)$$

j $\int \frac{2x^3+x^2-5}{x^2-1}\,dx$

$$\frac{2x^3+x^2-5}{x^2-1} = \frac{2x^3-2x+x^2+2x-5}{x^2-1}$$
$$= \frac{2x(x^2-1)}{x^2-1}+\frac{x^2+2x-5}{x^2-1}$$
$$= 2x+\frac{x^2-1+2x-4}{x^2-1}$$
$$= 2x+\frac{x^2-1}{x^2-1}+\frac{2x-4}{x^2-1}$$
$$= 2x+1+\frac{2x-4}{(x+1)(x-1)}$$

Now $\frac{2x-4}{(x+1)(x-1)} = \frac{a}{x+1}+\frac{b}{x-1}$

$$= \frac{a(x-1)+b(x+1)}{(x+1)(x-1)}$$

So $2x-4 = a(x-1)+b(x+1)$

Let $x=-1$, $-2-4=-2a$

$-6=-2a$

$a=3$

Let $x=1$, $2-4=2b$

$-2=2b$

$b=-1$

So $\dfrac{2x-4}{(x+1)(x-1)} = \dfrac{3}{x+1} - \dfrac{1}{x-1}$

Therefore $\displaystyle\int \frac{2x^3+x^2-5}{x^2-1}\,dx$

$= \displaystyle\int \left(2x+1+\frac{3}{x+1}-\frac{1}{x-1}\right)dx$

$= x^2 + x + 3\log_e(x+1) - \log_e(x-1) + c$

$(x > 1)$

k $\displaystyle\int \frac{x^2-x+2}{x^2+2x+1}\,dx$

$\dfrac{x^2-x+2}{x^2+2x+1} = \dfrac{x^2+2x+1-3x+1}{x^2+2x+1}$

$= \dfrac{x^2+2x+1}{x^2+2x+1} + \dfrac{-3x+1}{x^2+2x+1}$

$= 1 + \dfrac{-3x+1}{(x+1)^2}$

Now $\dfrac{-3x+1}{(x+1)^2} = \dfrac{a}{(x+1)} + \dfrac{b}{(x+1)^2}$

$= \dfrac{a(x+1)+b}{(x+1)^2}$

So $-3x+1 = a(x+1)+b$

Let $x=-1$, $3+1 = b$

$b = 4$

Let $x = 0$, $1 = a + 4$

$a = -3$

So $\dfrac{-3x+1}{(x+1)^2} = \dfrac{-3}{x+1} + \dfrac{4}{(x+1)^2}$

Therefore $\displaystyle\int \frac{x^2-x+2}{x^2+2x+1}\,dx$

$= \displaystyle\int \left(1-\frac{3}{x+1}+\frac{4}{(x+1)^2}\right)dx$

$= \displaystyle\int \left(1-\frac{3}{x+1}+4(x+1)^{-2}\right)dx$

$= x - 3\log_e(x+1) - 4(x+1)^{-1} + c$

$(x > -1)$

$= x - 3\log_e(x+1) - \dfrac{4}{x+1} + c$

l $\displaystyle\int \frac{2x^2-9x+7}{x^2-6x+9}\,dx$

$\dfrac{2x^2-9x+7}{x^2-6x+9} = \dfrac{2x^2-12x+18+3x-11}{x^2-6x+9}$

$= \dfrac{2(x^2-6x+9)}{x^2-6x+9} + \dfrac{3x-11}{x^2-6x+9}$

$= 2 + \dfrac{3x-11}{(x-3)^2}$

Now $\dfrac{3x-11}{(x-3)^2} = \dfrac{a}{(x-3)} + \dfrac{b}{(x-3)^2}$

$= \dfrac{a(x-3)+b}{(x-3)^2}$

So $3x - 11 = a(x-3) + b$

Let $x = 3$, $9 - 11 = b$

$b = -2$

Let $x = 0$, $-11 = -3a - 2$

$-9 = -3a$

$a = 3$

So $\dfrac{3x-11}{(x-3)^2} = \dfrac{3}{(x-3)} - \dfrac{2}{(x-3)^2}$

Therefore $\displaystyle\int \frac{2x^2-9x+7}{x^2-6x+9}\,dx$

$= \displaystyle\int \left(2+\frac{3}{x-3}-\frac{2}{(x-3)^2}\right)dx$

$= \displaystyle\int \left(2+\frac{3}{x-3}-2(x-3)^{-2}\right)dx$

$= 2x + 3\log_e(x-3) + 2(x-3)^{-1} + c$

$(x > 3)$

$= 2x + 3\log_e(x-3) + \dfrac{2}{x-3} + c$

8 a $\displaystyle\int \frac{4-x}{x(x+2)}\,dx$

$\dfrac{4-x}{x(x+2)} = \dfrac{a}{x} + \dfrac{b}{x+2}$

$= \dfrac{a(x+2)+bx}{x(x+2)}$

So $4 - x = a(x+2) + bx$

Let $x = -2$, $6 = -2b$

$b = -3$

Let $x = 0$, $4 = 2a$

$a = 2$

Therefore $\displaystyle\int \frac{4-x}{x(x+2)}\,dx$

$= \displaystyle\int \left(\frac{2}{x}-\frac{3}{x+2}\right)dx$

$= 2\log_e(x) - 3\log_e(x+2) + c$

$(x > 0)$

b $\displaystyle\int \frac{9x+8}{(x-3)(x+4)}\,dx$

$\dfrac{9x+8}{(x-3)(x+4)} = \dfrac{a}{x-3} + \dfrac{b}{x+4}$

$= \dfrac{a(x+4)+b(x-3)}{(x-3)(x+4)}$

So $9x + 8 = a(x+4) + b(x-3)$

Let $x = -4$, $-36 + 8 = -7b$

$-28 = -7b$

$b = 4$

Let $x = 3$, $27 + 8 = 7a$

$35 = 7a$

$a = 5$

Therefore $\displaystyle\int \frac{9x+8}{(x-3)(x+4)}\,dx$

$= \displaystyle\int \left(\frac{5}{x-3}+\frac{4}{x+4}\right)dx$

$= 5\log_e(x-3) + 4\log_e(x+4) + c$

$(x > 3)$

c $\displaystyle\int \frac{5(x+1)}{x^2-25}\,dx$

$= \displaystyle\int \frac{(5x+5)}{x^2-25}\,dx$

$\dfrac{5x+5}{x^2-25} = \dfrac{5x+5}{(x-5)(x+5)}$

$= \dfrac{a}{x-5} + \dfrac{b}{x+5}$

$= \dfrac{a(x+5)+b(x-5)}{(x-5)(x+5)}$

So $5x + 5 = a(x+5) + b(x-5)$

Let $x = -5$, $-25 + 5 = -10b$

$-20 = -10b$

$b = 2$

Let $x = 5$, $25 + 5 = 10a$

$30 = 10a$

$a = 3$

Therefore $\int \frac{5(x+1)}{x^2 - 25}\,dx$

$$= \int \left(\frac{3}{x-5} + \frac{2}{x+5} \right) dx$$

$= 3\log_e(x - 5) + 2\log_e(x + 5) \quad (x > 5)$

d $\int \frac{x^2+3}{x^2-9}\,dx$

$$\frac{x^2+3}{x^2-9} = \frac{x^2-9+12}{x^2-9}$$

$$= \frac{x^2-9}{x^2-9} + \frac{12}{x^2-9}$$

$$= 1 + \frac{12}{(x-3)(x+3)}$$

Now $\frac{12}{(x-3)(x+3)} = \frac{a}{x-3} + \frac{b}{x+3}$

$$= \frac{a(x+3)+b(x-3)}{(x+3)(x-3)}$$

So $12 = a(x + 3) + b(x - 3)$

Let $x = -3$, $12 = -6b$

$b = -2$

Let $x = 3$, $12 = 6a$

$a = 2$

So $\frac{12}{(x-3)(x+3)} = \frac{2}{x-3} - \frac{2}{x+3}$

Therefore $\int \frac{x^2+3}{x^2-9}\,dx$

$$= \int \left(1 + \frac{2}{x-3} - \frac{2}{x+3} \right) dx$$

$= x + 2\log_e(x - 3) - 2\log_e(x + 3) + c$

$(x > 3)$

e $\int \frac{x^2+3x-4}{(x-4)(x+2)}\,dx$

$$\frac{x^2+3x-4}{(x-4)(x+2)} = \frac{x^2+3x-4}{x^2-2x-8}$$

$$= \frac{x^2-2x-8+5x+4}{x^2-2x-8}$$

$$= \frac{x^2-2x-8}{x^2-2x-8} + \frac{5x+4}{x^2-2x-8}$$

$$= 1 + \frac{5x+4}{(x-4)(x+2)}$$

Now $\frac{5x+4}{(x-4)(x+2)} = \frac{a}{x-4} + \frac{b}{x+2}$

$$= \frac{a(x+2)+b(x-4)}{(x-4)(x+2)}$$

So $5x + 4 = a(x + 2) + b(x - 4)$

Let $x = -2$, $-10 + 4 = -6b$

$-6 = -6b$

$b = 1$

Let $x = 4$, $20 + 4 = 6a$

$24 = 6a$

$a = 4$

So $\frac{5x+4}{(x-4)(x+2)} = \frac{4}{x-4} + \frac{1}{x+2}$

Therefore $\int \frac{x^2+3x-4}{(x-4)(x+2)}\,dx$

$$= \int \left(1 + \frac{4}{x-4} + \frac{1}{x+2} \right) dx$$

$= x + 4\log_e(x - 4) + \log_e(x + 2) + c \quad (x > 4)$

f $\int \frac{x^2+4x+1}{x^2+6x-7}\,dx$

$$\frac{x^2+4x+1}{x^2+6x-7} = \frac{x^2+6x-7-2x+8}{x^2-6x-7}$$

$$= \frac{x^2+6x-7}{x^2+6x-7} + \frac{-2x+8}{x^2+6x-7}$$

$$= 1 + \frac{-2x+8}{(x+7)(x-1)}$$

Now $\frac{-2x+8}{(x+7)(x-1)} = \frac{a}{x+7} + \frac{b}{x-1}$

$$= \frac{a(x-1)+b(x+7)}{(x+7)(x-1)}$$

So $-2x + 8 = a(x - 1) + b(x + 7)$

Let $x = -7$, $14 + 8 = -8a$

$22 = -8a$

$a = \frac{-11}{4}$

Let $x = 1$, $-2 + 8 = 8b$

$6 = 8b$

$b = \frac{3}{4}$

So $\frac{-2x+8}{(x+7)(x-1)} = \frac{-11}{4(x+7)} + \frac{3}{4(x-1)}$

Therefore $\int \frac{x^2+4x+1}{x^2+6x-7}\,dx$

$$= \int \left(1 - \frac{11}{4(x+7)} + \frac{3}{4(x-1)} \right) dx$$

$= x - \frac{11}{4}\log_e(x + 7) + \frac{3}{4}\log_e(x - 1) + c$

$(x > 1)$

g $\int \frac{x^3+x^2-4x}{x^2-4x+4}\,dx$

$$\frac{x^3+x^2-4x}{x^2-4x+4} = \frac{x^3-4x^2+4x+5x^2-8x}{x^2-4x+4}$$

$$= \frac{x(x^2-4x+4)}{x^2-4x+4} + \frac{5x^2-8x}{x^2-4x+4}$$

$$= x + \frac{5x^2-20x+20+12x-20}{x^2-4x+4}$$

$$= x + \frac{5(x^2-4x+4)}{x^2-4x+4} + \frac{12x-20}{x^2-4x+4}$$

$$= x + 5 + \frac{12x-20}{(x-2)^2}$$

Now $\frac{12x-20}{(x-2)^2} = \frac{a}{(x-2)^2} + \frac{b}{(x-2)}$

$$= \frac{a+b(x-2)}{(x-2)^2}$$

So $12x - 20 = a + b(x - 2)$

Let $x = 2$, $24 - 20 = a$

$a = 4$

Let $x = 0$, $-20 = 4 - 2b$

$-24 = -2b$

$b = 12$

So $\dfrac{12x-20}{(x-2)^2} = \dfrac{4}{(x-2)^2} + \dfrac{12}{x-2}$

Therefore $\displaystyle\int \frac{x^3+x^2-4x}{x^2-4x+4}\,dx$

$= \displaystyle\int \left(x+5+\frac{4}{(x-2)^2}+\frac{12}{x-2} \right) dx$

$= \displaystyle\int \left(x+5+4(x-2)^{-2}+\frac{12}{x-2} \right) dx$

$= \dfrac{1}{2}x^2 + 5x - 4(x-2)^{-1} + 12\log_e(x-2) + c \quad (x > 2)$

$= \dfrac{1}{2}x^2 + 5x - \dfrac{4}{x-2} + 12\log_e(x-2) + c$

h $\displaystyle\int \frac{4x^2+6x-4}{2x^2-x-6}\,dx$

$\dfrac{4x^2+6x-4}{2x^2-x-6} = \dfrac{4x^2-2x-12+8x+8}{2x^2-x-6}$

$= \dfrac{2(2x^2-x-6)}{2x^2-x-6} + \dfrac{8x+8}{2x^2-x-6}$

$= 2 + \dfrac{8x+8}{(2x+3)(x-2)}$

Now $\dfrac{8x+8}{(2x+3)(x-2)} = \dfrac{a}{2x+3} + \dfrac{b}{x-2}$

$= \dfrac{a(x-2)+b(2x+3)}{(2x+3)(x-2)}$

So $8x + 8 = a(x-2) + b(2x+3)$

Let $x = \dfrac{-3}{2}$, $\dfrac{-24}{2} + 8 = \dfrac{-7}{2}a$

$-12 + 8 = \dfrac{-7}{2}a$

$-4 = \dfrac{-7}{2}a$

$a = \dfrac{8}{7}$

Let $x = 2$, $16 + 8 = (4+3)b$

$24 = 7b$

$b = \dfrac{24}{7}$

So $\dfrac{8x+8}{(2x+3)(x-2)} = \dfrac{8}{7(2x+3)} + \dfrac{24}{7(x-2)}$

Therefore $\displaystyle\int \frac{4x^2+6x-4}{2x^2-x-6}\,dx$

$= \displaystyle\int \left(2+\frac{8}{7(2x+3)}+\frac{24}{7(x-2)} \right) dx$

$= 2x + \dfrac{8}{7\times 2}\log_e(2x+3) + \dfrac{24}{7}\log_e(x-2) + c \quad (x > 2)$

$= 2x + \dfrac{4}{7}\log_e(2x+3) + \dfrac{24}{7}\log_e(x-2) + c$

i $\displaystyle\int \frac{x+1}{x^2+4}\,dx$

$= \displaystyle\int \left(\frac{x}{x^2+4} + \frac{1}{x^2+4} \right) dx$

$= \displaystyle\int \frac{x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$

Let $u = x^2 + 4$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

So $\int \frac{x}{x^2+4} dx + \int \frac{1}{x^2+4} dx$

$= \int \frac{x}{u} \times \frac{du}{2x} + \int \frac{1}{x^2+4} dx$

$= \frac{1}{2}\int \frac{1}{u} du + \frac{1}{2}\int \frac{2}{2^2+x^2} dx$

$= \frac{1}{2}\log_e(u) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

$= \frac{1}{2}\log_e(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

j $\int \frac{4x-2}{x^2+9} dx$

$= \int \left(\frac{4x}{x^2+9} - \frac{2}{x^2+9}\right) dx$

$= \int \frac{4x}{x^2+9} dx - \int \frac{2}{x^2+9} dx$

Let $u = x^2 + 9$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

So $\int \frac{4x}{x^2+9} dx - \int \frac{2}{x^2+9} dx$

$= \int \frac{4x}{u} \times \frac{du}{2x} - 2\int \frac{1}{x^2+9} dx$

$= 2\int \frac{1}{u} du - \frac{2}{3}\int \frac{3}{3^2+x^2} dx$

$= 2\log_e(u) - \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$

$= 2\log_e(x^2+9) - \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$

k $\int \frac{5x^2+2x+17}{(x-1)(x+2)(x-3)} dx$

$$\frac{5x^2+2x+17}{(x-1)(x+2)(x-3)} = \frac{a}{(x-1)} + \frac{b}{(x+2)} + \frac{c}{(x-3)}$$

$$= \frac{a(x+2)(x-3) + b(x-1)(x-3) + c(x-1)(x+2)}{(x-1)(x+2)(x-3)}$$

So $5x^2 + 2x + 17 = a(x+2)(x-3) + b(x-1)(x-3) + c(x-1)(x+2)$

Let $x = -2$, $5 \times 4 - 4 + 17 = (-3) \times (-5)b$

$20 - 4 + 17 = 15b$

$33 = 15b$

$b = \frac{33}{15}$

Let $x = 1$, $5 + 2 + 17 = 2 \times (-3)a$

$24 = -6a$

$a = -4$

Let $x = 3$, $5 \times 9 + 6 + 17 = 2 \times 5c$

$45 + 23 = 10c$

$68 = 10c$

$c = \frac{34}{5}$

So $\int \frac{5x^2+2x+17}{(x-1)(x+2)(x-3)} dx$

$= \int \left(\frac{-4}{x-1} + \frac{33}{15(x+2)} + \frac{34}{5(x-3)}\right) dx$

$= -4\log_e(x-1) + \frac{33}{15}\log_e(x+2) + \frac{34}{5}\log_e(x-3) + c \quad (x > 3)$

l $\int \frac{x^2+18x+5}{(x+1)(x-2)(x+3)}\,dx$

$$\frac{x^2+18x+5}{(x+1)(x-2)(x+3)} = \frac{a}{x+1}+\frac{b}{x-2}+\frac{c}{x+3}$$

$$= \frac{a(x-2)(x+3)+b(x+1)(x+3)+c(x+1)(x-2)}{(x+1)(x-2)(x+3)}$$

So $x^2 + 18x + 5 = a(x - 2)(x + 3) + b(x + 1)(x + 3) + c(x + 1)(x - 2)$

Let $x = -3$, $9 - 3 \times 18 + 5 = (-2) \times (-5)c$

$9 - 54 + 5 = 10c$

$-40 = 10c$

$c = -4$

Let $x = -1$, $1 - 18 + 5 = -3 \times 2a$

$-12 = -6a$

$a = 2$

Let $x = 2$, $4 + 36 + 5 = 3 \times 5b$

$45 = 15b$

$b = 3$

Therefore $\int \frac{x^2+8x+5}{(x+1)(x-2)(x+3)}\,dx$

$= \int\left(\frac{2}{x+1}+\frac{3}{x-2}-\frac{4}{x+3}\right)dx$

$= 2\log_e(x+1)+3\log_e(x-2)-4\log_e(x+3)+c \quad (x > 2)$

m $\int \frac{x^2+8x+9}{(x-1)(x+2)^2}\,dx$

$$\frac{x^2+8x+9}{(x-1)(x+2)^2} = \frac{a}{x-1}+\frac{b}{x+2}+\frac{c}{(x+2)^2}$$

$$= \frac{a(x+2)^2+b(x-1)(x+2)+c(x-1)}{(x-1)(x+2)^2}$$

So $x^2 + 8x + 9 = a(x + 2)^2 + b(x - 1)(x + 2) + c(x - 1)$

Let $x = -2$, $4 - 16 + 9 = -3c$

$-3 = -3c$

$c = 1$

Let $x = 1$, $1 + 8 + 9 = 3^2a$

$18 = 9a$

$a = 2$

Let $x = 0$, $9 = 2(2)^2 - 1 \times 2b + 1 \times (-1)$

$9 = 8 - 2b - 1$

$2 = -2b$

$b = -1$

Therefore $\int \frac{x^2+8x+9}{(x-1)(x+2)^2}\,dx$

$= \int\left(\frac{2}{x-1}-\frac{1}{x+2}+\frac{1}{(x+2)^2}\right)dx$

$= \int\left(\frac{2}{x-1}-\frac{1}{x+2}+(x+2)^{-2}\right)dx$

$= 2\log_e(x-1)-\log_e(x+2)-(x+2)^{-1}+c \quad (x > 1)$

$= 2\log_e(x-1)-\log_e(x+2)-\frac{1}{x+2}+c$

n $\int \frac{x^2+5x+1}{(x^2+1)(2-x)}\,dx$

$$\frac{x^2+5x+1}{(x^2+1)(2-x)} = \frac{ax+b}{(x^2+1)}+\frac{c}{2-x}$$, since $x^2 + 1$ cannot be factorised

$$= \frac{(ax+b)(2-x)+c(x^2+1)}{(x^2+1)(2-x)}$$

So $x^2 + 5x + 1 = (ax + b)(2 - x) + c(x^2 + 1)$

Let $x = 2$, $4 + 10 + 1 = 5c$

$15 = 5c$

$c = 3$

Let $x = 0$, $1 = 2b + 3$
$-2 = 2b$
$b = -1$

Let $x = 1$, $1 + 5 + 1 = (a - 1)(2 - 1) + 3(1 + 1)$
$7 = a - 1 + 6$
$a = 2$

So $\int \frac{x^2+5x+1}{(x^2+1)(2-x)}dx$

$= \int\left(\frac{2x-1}{x^2+1} - \frac{3}{2-x}\right)dx$

$= \int\left(\frac{2x}{x^2+1} - \frac{1}{x^2+1} - \frac{3}{2-x}\right)dx$

$= \int \frac{2x}{x^2+1}dx - \int \frac{1}{x^2+1}dx + \int \frac{3}{2-x}dx$

Let $u = x^2 + 1$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

So $\int \frac{2x}{x^2+1}dx - \int \frac{1}{x^2+1}dx - \int \frac{3}{2-x}dx$

$= \int \frac{2x}{u} \times \frac{du}{2x} - \int \frac{1}{x^2+1}dx - 3\int \frac{1}{2-x}dx$

$= \int \frac{1}{u}du - \int \frac{1}{x^2+1}dx - 3\int \frac{1}{2-x}dx$

$= \log_e(u) - \tan^{-1}(x) + 3\log_e(2-x) + c \quad (x < 2)$

$= \log_e(x^2+1) - \tan^{-1}(x) + 3\log_e(2-x) + c$

9 a If $f'(x) = \frac{6}{x^2-1}$ and $f(2) = 3\log_e(2)$, find $f(x)$

$f'(x) = \frac{6}{x^2-1}$

$f(x) = \int \frac{6}{x^2-1}dx$

$\frac{6}{x^2-1} = \frac{6}{(x-1)(x+1)}$

$= \frac{a}{x-1} + \frac{b}{x+1}$

$= \frac{a(x+1)+b(x-1)}{(x-1)(x+1)}$

So $6 = a(x + 1) + b(x - 1)$

Let $x = -1$, $6 = -2b$
$b = -3$

Let $x = 1$, $6 = 2a$
$a = 3$

So $\int \frac{6}{x^2-1}dx$

$= \int\left(\frac{3}{x-1} - \frac{3}{x+1}\right)dx$

$= 3\int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx$

$= 3\left[\log_e(x-1) - \log_e(x+1)\right] + c$

$f(x) = 3\log_e\left(\frac{x-1}{x+1}\right) + c$

$f(2) = 3\log_e\left(\frac{2-1}{2+1}\right) + c = 3\log_e(2)$

$3\log_e\left(\frac{1}{3}\right) + c = 3\log_e(2)$

$c = 3\left(\log_e(2) - \log_e\left(\frac{1}{3}\right)\right)$

$c = 3\log_e(2 \times 3)$
$c = 3\log_e(6)$

Therefore $f(x) = 3\log_e\left(\dfrac{x-1}{x+1}\right) + 3\log_e(6)$

$$f(x) = 3\left[\log_e\left(\frac{x-1}{x+1}\right) + \log_e(6)\right]$$

$$f(x) = 3\log_e\left[\frac{6(x-1)}{x+1}\right]$$

b For $3\log_e\left[\dfrac{6(x-1)}{x+1}\right]$ to exist

$\dfrac{6(x-1)}{x+1} > 0$ and $x + 1 \neq 0$

$x - 1 > 0$

$x > 1$ and $x \neq -1$

Therefore the domain of $f(x)$ is $x > 1$

10 a Find $g(x)$ if $g'(x) = \dfrac{x^2+1}{x^2-2x-3}$ and $g(4) = 4 - \log_e(5)$

$$g'(x) = \frac{x^2+1}{x^2-2x-3}$$

$$g(x) = \int \frac{x^2+1}{x^2-2x-3}\,dx$$

$$\frac{x^2+1}{x^2-2x-3} = \frac{x^2-2x-3+2x+4}{x^2-2x-3}$$

$$= \frac{x^2-2x-3}{x^2-2x-3} + \frac{2x+4}{x^2-2x-3}$$

$$= 1 + \frac{2x+4}{(x-3)(x+1)}$$

Now $\dfrac{2x+4}{(x-3)(x+1)} = \dfrac{a}{x-3} + \dfrac{b}{x+1}$

$$= \frac{a(x+1)+b(x-3)}{(x-3)(x+1)}$$

So $2x + 4 = a(x + 1) + b(x - 3)$

Let $x = -1$, $-2 + 4 = -4b$

$2 = -4b$

$b = \dfrac{-1}{2}$

Let $x = 3$, $6 + 4 = 4a$

$10 = 4a$

$a = \dfrac{5}{2}$

So $\dfrac{2x+4}{(x-3)(x+1)} = \dfrac{5}{2(x-3)} - \dfrac{1}{2(x+1)}$

Therefore $g(x) = \displaystyle\int\left(1 + \frac{5}{2(x-3)} - \frac{1}{2(x+1)}\right)dx$

$$= x + \frac{5}{2}\log_e(x-3) - \frac{1}{2}\log_e(x+1) + c$$

$$= x + \frac{1}{2}\left\{\log_e\left[(x-3)^5\right] - \log_e(x+1)\right\} + c$$

$$= x + \frac{1}{2}\log_e\left[\frac{(x-3)^5}{x+1}\right] + c$$

$$g(4) = 4 + \frac{1}{2}\log_e\left[\frac{(4-3)^5}{4+1}\right] + c = 4 - \log_e(5)$$

$$\cancel{4} + \frac{1}{2}\log_e\left[\frac{(4-3)^5}{4+1}\right] + c = \cancel{4} - \log_e(5)$$

$$\frac{1}{2}\log_e\left(\frac{1}{5}\right)+c=-\log_e(5)$$

$$c=\frac{-1}{2}\log_e\left(\frac{1}{5}\right)-\log_e(5)$$

$$c=\frac{1}{2}\log_e(5)-\log_e(5)$$

$$c=\frac{-1}{2}\log_e(5)$$

Therefore $g(x)=x+\frac{1}{2}\log_e\left[\frac{(x-3)^5}{x+1}\right]-\frac{1}{2}\log_e(5)$

$$g(x)=x+\frac{1}{2}\left\{\log_e\left[\frac{(x-3)^5}{x+1}\right]-\log_e(5)\right\}$$

$$g(x)=x+\frac{1}{2}\log_e\left[\frac{(x-3)^5}{5(x+1)}\right]$$

b For $\log_e\left[\frac{(x-3)^5}{5(x+1)}\right]$ to exist

$\frac{(x-3)^5}{5(x+1)}>0$ and $x+1\neq 0$

$x-3>0$

$x>3$ and $x\neq -1$

Therefore the domain of $g(x)$ is $x>3$.

Exercise 4F — Integration by parts

1 a $\int x\cos(x)\,dx$

$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$

Let $u=x$ and $\frac{dv}{dx}=\cos(x)$

So $\frac{du}{dx}=1$ and $v=\sin(x)$

$$\int x\cos(x)dx=x\sin(x)-\int\sin(x)dx$$
$$=x\sin(x)+\cos(x)+c$$

b $\int xe^x\,dx$

$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$

Let $u=x$ and $\frac{dv}{dx}=e^x$

So $\frac{du}{dx}=1$ and $v=e^x$

$$\int xe^x\,dx=xe^x-\int e^x\,dx$$
$$=xe^x-e^x+c$$
$$=(x-1)e^x+c$$

c $\int x\log_e(x)\,dx$

$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$

Let $u=\log_e(x)$ and $\frac{dv}{dx}=x$

So $\frac{du}{dx}=\frac{1}{x}$ and $v=\frac{1}{2}x^2$

$$\int x\log_e(x)dx=\log_e(x)\times\frac{1}{2}x^2-\int\frac{1}{2}x^2\frac{1}{x}dx$$
$$=\frac{1}{2}x^2\log_e(x)-\frac{1}{2}\int x\,dx$$
$$=\frac{1}{2}x^2\log_e(x)-\frac{1}{4}x^2+c$$

d $\int x\sin(2x)\,dx$

$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$

Let $u=x$ and $\frac{dv}{dx}=\sin(2x)$

So $\frac{du}{dx}=1$ and $v=\frac{-1}{2}\cos(2x)$

$$\int x\sin(2x)dx=\int x\left(\frac{-1}{2}\cos(2x)\right)-\int\left(\frac{-1}{2}\cos(2x)\right)dx$$
$$=\frac{-1}{2}x\cos(2x)+\frac{1}{2}\int\cos(2x)dx$$
$$=\frac{-1}{2}x\cos(2x)+\frac{1}{4}\sin(2x)+c$$
$$=\frac{1}{4}\sin(2x)-\frac{1}{2}x\cos(2x)+c$$

2 a $\int x^2\cos(x)\,dx$

$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$

Let $u=x^2$ and $\frac{dv}{dx}=\cos(x)$

So $\frac{du}{dx}=2x$ and $v=\sin(x)$

$$\int x^2\cos(x)dx=x^2\sin(x)-\int\sin x(2x)dx$$
$$=x^2\sin(x)-2\int x\sin(x)dx$$

Let $u=x$ and $\frac{dv}{dx}=\sin(x)$

So $\frac{du}{dx}=1$ and $v=-\cos(x)$

$$\int x^2\cos(x)dx=x^2\sin(x)-2(-\cos x(x)$$
$$-\int(-\cos(x))dx)$$
$$=x^2\sin(x)+2x\cos(x)-2\int\cos(x)dx$$
$$=x^2\sin(x)+2x\cos(x)-2\sin(x)+c$$

b $\int x^2\sin(x)\,dx$

$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$

Let $u=x^2$ and $\frac{dv}{dx}=\sin(x)$

So $\frac{du}{dx}=2x$ and $v=-\cos(x)$

$$\int x^2\sin(x)dx=x^2(-\cos(x))-\int(-\cos(x)(2x))dx$$
$$=-x^2\cos(x)+2\int x\cos(x)dx$$

Let $u=x$ and $\frac{dv}{dx}=\cos(x)$

So $\frac{du}{dx}=1$ and $v=\sin(x)$

$$\int x^2\sin(x)dx=-x^2\cos(x)+2(x\sin(x)-\int\sin(x)dx)$$
$$=-x^2\cos(x)+2x\sin(x)+2\cos(x)+c$$

c $\int x^2e^{2x}\,dx$

$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$

Let $u=x^2$ and $\frac{dv}{dx}=e^{2x}$

So $\frac{du}{dx}=2x$ and $v=\frac{1}{2}e^{2x}$

$$\int x^2 e^{2x}\,dx = x^2 \times \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}(2x)\,dx$$

$$= \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx$$

Let $u = x$ and $\frac{dv}{dx} = e^{2x}$

So $\frac{du}{dx} = 1$ and $v = \frac{1}{2}e^{2x}$

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - (x \times \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}dx)$$

$$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{2}\int e^{2x}dx$$

$$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + c$$

d $\int x^2 (x+1)^5\,dx$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Let $u = x^2$ and $\frac{dv}{dx} = (x+1)^5$

So $\frac{du}{dx} = 2x$ and $v = \frac{1}{6}(x+1)^6$

$$\int x^2 (x+1)^5 dx = x^2 \times \frac{1}{6}(x+1)^6 - \int \frac{1}{6}(x+1)^6 (2x)\,dx$$

$$= \frac{1}{6}x^2 (x+1)^6 - \frac{1}{3}\int x(x+1)^6\,dx$$

Let $u = x$ and $\frac{dv}{dx} = (x+1)^6$

So $\frac{du}{dx} = 1$ and $v = \frac{1}{7}(x+1)^7$

$$\int x^2 (x+1)^5 dx = \frac{1}{6}x^2 (x+1)^6 - \frac{1}{3}\left(x \times \frac{1}{7}(x+1)^7 - \int \frac{1}{7}(x+1)^7\,dx\right)$$

$$= \frac{1}{6}x^2 (x+1)^6 - \frac{1}{21}x(x+1)^7 + \frac{1}{21}\int (x+1)^7\,dx$$

$$= \frac{1}{6}x^2 (x+1)^6 - \frac{1}{21}x(x+1)^7 + \frac{1}{21}\left[\frac{1}{8}(x+1)^8\right] + c$$

$$= \frac{1}{6}x^2 (x+1)^6 - \frac{1}{21}x(x+1)^7 + \frac{1}{168}(x+1)^8 + c$$

3 a $\int \log_e(2x)\,dx$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Let $u = \log_e(2x)$ and $\frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{x}$ and $v = x$

$$\int \log_e(2x)\,dx = (\log_e(2x))\,x - \int x\frac{1}{x}\,dx$$

$$= x\log_e(2x) - \int 1\,dx$$

$$= x\log_e(2x) - x + \text{c}$$

b $\int \sin^{-1}(x)\,dx$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Let $u = \sin^{-1}(x)$ and $\frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ and $v = x$

$$\int \sin^{-1}(x)\,dx = (\sin^{-1}(x))x - \int x\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Let $u = 1 - x^2$

$$\frac{du}{dx} = -2x \quad \text{or} \quad dx = \frac{du}{-2x}$$

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{u}} \times \frac{du}{-2x}$$

$$= x\sin^{-1}(x) + \frac{1}{2}\int u^{\frac{-1}{2}}\,du$$

$$= x\sin^{-1}(x) + \frac{1}{2}\left(2u^{\frac{1}{2}}\right) + c$$

$$= x\sin^{-1}(x) + \sqrt{u} + c$$

$$= x\sin^{-1}(x) + \sqrt{1-x^2} + c$$

4 a $\int e^x \cos(x)\,dx$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Let $u = \cos(x)$ and $\frac{dv}{dx} = e^x$

So $\frac{du}{dx} = -\sin(x)$ and $v = e^x$

$$\int e^x \cos(x) = \cos(x)\,e^x - \int e^x(-\sin(x))\,dx$$

$$= e^x\cos(x) + \int e^x \sin(x)\,dx$$

Let $u = \sin(x)$ and $\frac{dv}{dx} = e^x$

So $\frac{du}{dx} = \cos(x)$ and $v = e^x$

$$\int e^x\cos(x)\,dx = e^x\cos(x) + \sin(x)e^x - \int e^x\cos(x)\,dx$$

$$2\int e^x\cos(x)\,dx = e^x\cos(x) + e^x\sin(x)$$

$$\int e^x\cos(x)\,dx = \frac{1}{2}(e^x\cos(x) + e^x\sin(x)) + c$$

b $\int e^{2x}\cos(x)\,dx$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Let $u = \cos(x)$ and $\frac{dv}{dx} = e^{2x}$

So $\frac{du}{dx} = -\sin(x)$ and $v = \frac{1}{2}e^{2x}$

$$\int e^{2x}\cos(x)\,dx = \cos(x)\times\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}(-\sin(x))\,dx$$

$$= \frac{1}{2}e^{2x}\cos(x) + \frac{1}{2}\int e^{2x}\sin(x)\,dx$$

Let $u = \sin(x)$ and $\frac{dv}{dx} = e^{2x}$

So $\frac{du}{dx} = \cos(x)$ and $v = \frac{1}{2}e^{2x}$

$$\int e^{2x}\cos(x)\,dx = \frac{1}{2}e^{2x}\cos(x) + \frac{1}{2}\left(\sin(x)\times\frac{1}{2}e^{2x} - \int\frac{1}{2}e^{2x}\cos(x)\,dx\right)$$

$$\int e^{2x}\cos(x)\,dx = \frac{1}{2}e^{2x}\cos(x) + \frac{1}{4}e^{2x}\sin(x) - \frac{1}{4}\int e^{2x}\cos(x)\,dx$$

$$\frac{5}{4}\int e^{2x}\cos(x)\,dx = \frac{1}{2}e^{2x}\cos(x) + \frac{1}{4}e^{2x}\sin(x)$$

$$\int e^{2x}\cos(x)\,dx = \frac{4}{5}\times\frac{1}{4}e^{2x}(2\cos(x)+\sin(x))+c$$
$$= \frac{e^{2x}}{5}(2\cos(x)+\sin(x))+c$$

c $\int e^{x}\sin(2x)\,dx$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Let $u=\sin(2x)$ and $\frac{dv}{dx}=e^x$

So $\frac{du}{dx}=2\cos(2x)$ and $v=e^x$

$$\int e^x\sin(2x)\,dx = (\sin(2x))e^x - \int e^x\times 2\cos(2x)\,dx$$
$$= e^x\sin(2x) - 2\int e^x\cos(2x)\,dx$$

Let $u=\cos(2x)$ and $\frac{dv}{dx}=e^x$

So $\frac{du}{dx}=-2\sin(2x)$ and $v=e^x$

$$\int e^x\sin(2x)\,dx = e^x\sin(2x)-2\left[(\cos(2x))e^x - \int e^x(-2\sin(2x))\,dx\right]$$
$$\int e^x\sin(2x)\,dx = e^x\sin(2x)-2e^x\cos(2x)-4\int e^x\sin(2x)\,dx$$
$$5\int e^x\sin(2x)\,dx = e^x\sin(2x)-2e^x\cos(2x)$$
$$\int e^x\sin(2x)\,dx = \frac{1}{5}e^x(\sin(2x)-2\cos(2x))+c$$

5 a $\int (x-1)\sin(x)\,dx$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Let $u=x-1$ and $\frac{dv}{dx}=\sin(x)$

So $\frac{du}{dx}=1$ and $v=-\cos(x)$

$$\int (x-1)\sin(x)\,dx = (x-1)(-\cos(x)) - \int(-\cos(x))\,dx$$
$$= -x\cos(x)+\cos(x)+\int\cos(x)\,dx$$
$$= -x\cos(x)+\cos(x)+\sin(x)+c$$

b $\int (x+2)e^x\,dx$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Let $u=x+2$ and $\frac{dv}{dx}=e^x$

So $\frac{du}{dx}=1$ and $v=e^x$

$$\int (x+2)e^x\,dx = (x+2)e^x - \int e^x\,dx$$
$$= xe^x+2e^x-e^x+c$$
$$= xe^x+e^x+c$$
$$= (x+1)e^x+c$$

c $\int x^3\log_e(x)\,dx$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Let $u=\log_e(x)$ and $\frac{dv}{dx}=x^3$

So $\frac{du}{dx}=\frac{1}{x}$ and $v=\frac{1}{4}x^4$

$$\int x^3\log_e(x)\,dx = \log_e(x)\times\frac{1}{4}x^4 - \int\frac{1}{4}x^4\times\frac{1}{x}dx$$
$$= \frac{1}{4}x^4\log_e(x)-\frac{1}{4}\int x^3\,dx$$

$= \frac{1}{4}x^4\log_e(x) - \frac{1}{4}\left(\frac{1}{4}x^4\right) + c$

$= \frac{1}{4}x^4\log_e(x) - \frac{1}{16}x^4 + c$

d $\int (x+1)\log_e(x)\,dx$

$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Let $u = \log_e(x)$ and $\frac{dv}{dx} = x + 1$

So $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{1}{2}x^2 + x$

$\int (x+1)\log_e(x)\,dx = \log_e(x)\left(\frac{1}{2}x^2 + x\right) - \int\left(\frac{1}{2}x^2 + x\right)\frac{1}{x}dx$

$= \frac{1}{2}(x^2 + 2x)\log_e(x) - \int\left(\frac{1}{2}x + 1\right)dx$

$= \frac{1}{2}(x^2 + 2x)\log_e(x) - \frac{1}{4}x^2 - x + c$

$= \frac{1}{2}(x^2 + 2x + 1)\log_e(x) - \frac{1}{4}x^2 - x - \frac{1}{2}\log_e(x) + c$

$= \frac{1}{2}(x+1)^2\log_e(x) - \frac{1}{4}x^2 - x - \frac{1}{2}\log_e(x) + c$

e $\int x^2\sin^{-1}(x)\,dx$

$= \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Let $u = \sin^{-1}(x)$ and $\frac{dv}{dx} = x^2$

So $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ and $v = \frac{1}{3}x^3$

$\int x^2\sin^{-1}(x)\,dx = \sin^{-1}(x)\times\frac{1}{3}x^3 - \int\frac{1}{3}x^3\times\frac{1}{\sqrt{1-x^2}}\,dx$

$= \frac{1}{3}x^3\sin^{-1}(x) - \frac{1}{3}\int\frac{x^3}{\sqrt{1-x^2}}\,dx$

Let $u = 1 - x^2$ and $x^2 = 1 - u$

$\frac{du}{dx} = -2x$ or $dx = \frac{du}{-2x}$

$\int x^2\sin^{-1}(x)\,dx = \frac{1}{3}x^3\sin^{-1}(x) - \frac{1}{3}\int\frac{x^3}{\sqrt{u}}\times\frac{du}{-2x}$

$= \frac{1}{3}x^3\sin^{-1}(x) + \frac{1}{6}\int\frac{x^2}{\sqrt{u}}du$

$= \frac{1}{3}x^3\sin^{-1}(x) + \frac{1}{6}\int\frac{1-u}{u^{\frac{1}{2}}}du$

$= \frac{1}{3}x^3\sin^{-1}(x) + \frac{1}{6}\int\left(u^{\frac{-1}{2}} - u^{\frac{1}{2}}\right)du$

$= \frac{1}{3}x^3\sin^{-1}(x) + \frac{1}{6}\left(2u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + c$

$= \frac{1}{3}x^3\sin^{-1}(x) + \frac{1}{3}u^{\frac{1}{2}} - \frac{1}{9}u^{\frac{3}{2}} + c$

$= \frac{1}{3}x^3\sin^{-1}(x) + \frac{1}{3}(1-x^2)^{\frac{1}{2}} - \frac{1}{9}(1-x^2)^{\frac{3}{2}} + c$

$= \frac{1}{3}x^3\sin^{-1}(x) + \frac{1}{9}(1-x^2)^{\frac{1}{2}}[3 - (1-x^2)] + c$

$= \frac{1}{3}x^3\sin^{-1}(x) + \frac{1}{9}(1-x^2)^{\frac{1}{2}}(3 - 1 + x^2) + c$

$= \frac{1}{3}x^3 \sin^{-1}(x) + \frac{1}{9}(1-x^2)^{\frac{1}{2}}(x^2+2) + c$

$= \frac{1}{3}x^3 \sin^{-1}(x) + \frac{1}{9}x^2\sqrt{1-x^2} + \frac{2}{9}\sqrt{1-x^2} + c$

f $\int (x+1)^2 e^x \, dx$

$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Let $u = (x+1)^2$ and $\frac{dv}{dx} = e^x$

So $\frac{du}{dx} = 2(x+1)$ and $v = e^x$

$\int (x+1)^2 e^x \, dx = (x+1)^2 e^x - \int 2(x+1)e^x \, dx$

$= (x+1)^2 e^x - 2\int (x+1)e^x \, dx$

Let $u = x+1$ and $\frac{dv}{dx} = e^x$

So $\frac{du}{dx} = 1$ and $v = e^x$

$\int (x+1)^2 e^x \, dx = (x+1)^2 e^x - 2\left[(x+1)e^x - \int e^x \, dx\right]$

$= (x+1)^2 e^x - 2(x+1)e^x + 2e^x + c$

$= (x^2 + 2x + 1 - 2x - 2 + 2)e^x + c$

$= (x^2+1)e^x + c$

g $\int x^3 e^x \, dx$

$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Let $u = x^3$ and $\frac{dv}{dx} = e^x$

So $\frac{du}{dx} = 3x^2$ and $v = e^x$

$\int x^3 e^x \, dx = x^3 e^x - \int e^x \times 3x^2 \, dx$

$= x^3 e^x - 3\int x^2 e^x \, dx$

Let $u = x^2$ and $\frac{dv}{dx} = e^x$

So $\frac{du}{dx} = 2x$ and $v = e^x$

$\int x^3 e^x \, dx = x^3 e^x - 3\int x^2 e^x - \int e^x \times 2x \, dx$

$= x^3 e^x - 3x^2 e^x + 6\int x e^x \, dx$

Let $u = x$ and $\frac{dv}{dx} = e^x$

So $\frac{du}{dx} = 1$ and $v = e^x$

$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6(e^x x - \int e^x \, dx)$

$= x^3 e^x - 3x^2 e^x + 6xe^x - 6\int e^x \, dx$

$= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c$

$= e^x(x^3 - 3x^2 + 6x - 6) + c$

h $\int \frac{x}{e^x} dx = \int x e^{-x} \, dx$

$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Let $u = x$ and $\frac{dv}{dx} = e^{-x}$

So $\frac{du}{dx} = 1$ and $v = -e^{-x}$

$\int \frac{x}{e^x} dx = x(-e^{-x}) - \int (-e^{-x}) \, dx$

$= -xe^{-x} + \int e^{-x}\,dx$

$= -xe^{-x} - e^{-x} + c$

$= -e^{-x}(x+1) + c$

i $\int x^n \log_e(x)\,dx$

$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}$

Let $u = \log_e(x)$ and $\frac{dv}{dx} = x^n$

So $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{1}{n+1}x^{n+1}$

$\int x^n \log_e(x)\,dx = \log_e(x) \times \frac{1}{n+1}x^{n+1} - \int \frac{1}{n+1}x^{n+1} \times \frac{1}{x}dx$

$= \frac{x^{n+1}}{n+1}\log_e(x) - \frac{1}{n+1}\int x^n\,dx$

$= \frac{x^{n+1}}{n+1}\log_e(x) - \frac{1}{n+1}\left(\frac{1}{n+1}x^{n+1}\right) + c$

$= \frac{x^{n+1}}{n+1}\left(\log_e(x) - \frac{1}{n+1}\right) + c$

6 a Show that $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Let $u = x^n$ and $\frac{dv}{dx} = e^x$

So $\frac{du}{dx} = nx^{n-1}$ and $v = e^x$

LHS $= \int x^n e^x\,dx$

$= x^n e^x - \int e^x \times nx^{n-1}\,dx$

$= x^n e^x - n\int x^{n-1}e^x\,dx$

= RHS, as required.

b $\int x^5 e^x\,dx = x^5e^x - 5\int x^4 e^x\,dx$

$= x^5e^x - 5(x^4e^x - 4\int x^3e^x\,dx)$

$= x^5e^x - 5x^4e^x + 20\int x^3e^x\,dx$

$= x^5e^x - 5x^4e^x + 20(x^3e^x - 3\int x^2e^x\,dx)$

$= x^5e^x - 5x^4e^x + 20x^3e^x - 60\int x^2e^x\,dx$

$= x^5e^x - 5x^4e^x + 20x^3e^x - 60(x^2e^x - 2\int xe^x\,dx)$

$= x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120\int xe^x\,dx$

$= x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120(xe^x - \int e^x\,dx)$

$= x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + c$

$= e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) + c$

7 a $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

LHS $= \int \cos^n(x)\,dx$

$= \int \cos(x)\cos^{n-1}(x)\,dx$

Let $u = \cos^{n-1}(x)$ and $\frac{dv}{dx} = \cos(x)$

So $\frac{du}{dx} = -(n-1)\cos^{n-2}(x)\sin(x)$ and $v = \sin(x)$

LHS $= \cos^{n-1}x\sin(x) - \int \sin(x)\left[-(n-1)\cos^{n-2}x\sin(x)\right]dx$

$= \sin(x)\cos^{n-1}(x) + (n-1)\int \sin^2(x)\cos^{n-2}(x)\,dx$

= RHS, as required

b $\int \cos^3(x)\,dx = \sin(x)\cos^2(x) + 2\int \sin^2(x)\cos(x)\,dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

$$\int \cos^3(x)\,dx = \sin(x)\cos^2(x) + 2\int u^2\cos(x)\frac{du}{\cos(x)}$$
$$= \sin(x)\cos^2(x) + 2\int u^2\,du$$
$$= \sin(x)\cos^2(x) + \frac{2}{3}u^3 + c$$
$$= \sin(x)\cos^2(x) + \frac{2}{3}\sin^3(x) + c$$

8 $\int \sin^6\frac{x}{2}\,dx = \int\left(\sin^2\frac{x}{2}\right)^3 dx$

$$= \int\left(\frac{1-\cos(x)}{2}\right)^3 dx$$
$$= \frac{1}{8}\int (1-\cos(x))^3\,dx$$
$$= \frac{1}{8}\int (1-3\cos(x)+3\cos^2(x)-\cos^3(x))\,dx$$
$$= \frac{1}{8}\left\{\int (1-3\cos(x))\,dx + 3\int \cos^2(x)\,dx - \int \cos^3(x)\,dx\right\}$$
$$= \frac{1}{8}\left\{x - 3\sin(x) + \frac{3}{2}\int (\cos 2x + 1)\,dx - \left(\sin(x)\cos^2(x) + \frac{2}{3}\sin^3(x)\right)\right\}$$, from **7(b)**
$$= \frac{1}{8}\left\{x - 3\sin(x) + \frac{3}{2}\left(\frac{1}{2}\sin(2x) + x\right) - \sin(x)\cos^2(x) - \frac{2}{3}\sin^3(x)\right\} + c$$
$$= \frac{1}{8}\left(x - 3\sin(x) + \frac{3}{4}\sin(2x) + \frac{3}{2}x - \sin(x)\cos^2(x) - \frac{2}{3}\sin^3(x)\right) + c$$
$$= \frac{1}{8}\left(x - 3\sin(x) + \frac{3}{2}\sin(x)\cos(x) + \frac{3}{2}x - \sin(x)\cos^2(x) - \frac{2}{3}\sin^3(x)\right) + c$$

Exercise 4G — Definite integrals

1 a i $\int_1^2 \frac{1}{9-x^2}\,dx$

The integrand exists if $9 - x^2 \neq 0$

$x^2 \neq 9$

$x \neq \pm 3$

The domain is $R\setminus\{-3, 3\}$

ii The integral does exist.

b i $\int_0^1 \frac{-1}{\sqrt{4-x^2}}\,dx$

The integral exists if $4 - x^2 > 0$

$x^2 < 4$

$-2 < x < 2$

The domain is $(-2, 2)$.

ii The integral does exist

c i $\int_3^5 \frac{dx}{\sqrt{16-x^2}}$

The integral exists if $16 - x^2 > 0$

$x^2 < 16$

$-4 < x < 4$

The domain is $(-4, 4)$

ii The integral does not exist.

d i $\int_{-1}^2 \frac{dx}{1+x^2}$

The integral exists if $1 + x^2 > 0$

$x^2 > -1$

This is true for all x, so the domain is R.

ii The integral does exist

e **i** $\int_1^4 \frac{2}{x}\,dx$

The integral exists if $x \neq 0$

The domain is $R\backslash\{0\}$

ii The integral does exist.

f **i** $\int_1^2 \frac{dx}{x(x+1)}$

The integral exists if $x(x+1) \neq 0$

$x \neq 0 \quad x+1 \neq 0$

$x \neq -1$

The domain is $R\backslash\{-1, 0\}$

ii The integral does exist

g **i** $\int_{-1}^1 \frac{4x+10}{x^2+5x+6}\,dx$

The integral exists if $x^2 + 5x + 6 \neq 0$

$(x+2)(x+3) \neq 0$

$x+2 \neq 0 \quad x+3 \neq 0$

$x \neq -2 \quad x \neq -3$

The domain is $R\backslash\{-2, -3\}$

ii The integral does exist

h **i** $\int_0^2 \frac{1}{(x-1)^2}\,dx$

The integral exists if $(x-1)^2 \neq 0$

$x - 1 \neq 0$

$x \neq 1$

The domain is $R\backslash\{1\}$

ii The integral does not exist

i **i** $\int_1^3 \frac{3x+2}{x^2-8x+12}\,dx$

The integral exists if $x^2 - 8x + 12 \neq 0$

$(x-6)(x-2) \neq 0$

$x \neq 6 \quad x \neq 2$

The domain is $R\backslash\{2, 6\}$

ii The integral does not exist

j **i** $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{4x^2+9}$

The integral exists if $4x^2 + 9 \neq 0$

$x^2 \neq \frac{-4}{9}$

This is true for all x, so the domain is R.

ii The integral does exist.

k **i** $\int_{-1}^0 \frac{dx}{\sqrt{1-9x^2}}$

The integral exists if $1 - 9x^2 > 0$

$x^2 < \frac{1}{9}$

$\frac{-1}{3} < x < \frac{1}{3}$

The domain is $\left(\frac{-1}{3}, \frac{1}{3}\right)$

ii The integral does not exist.

l **i** $\int_{\frac{1}{2}}^2 (2x-1)^{\frac{3}{2}}\,dx$

The integral exists if $2x - 1 \geq 0$

$x \geq \frac{1}{2}$

The domain is $\left[\frac{1}{2}, \infty\right]$

ii The integral does exist

m **i** $\int_0^2 \left(x + \frac{1}{x-2}\right)dx$

The integral exists if $x - 2 \neq 0$

$x \neq 2$

The domain is $R\backslash\{2\}$

ii The integral does not exist

n **i** $\int_0^3 (e^x + e^{-x})^2\,dx$

The integral exists for all values of x, so the domain is R

ii The integral does exist

2 **a** $\int_1^2 \frac{1}{9-x^2}\,dx$

$$\frac{1}{9-x^2} = \frac{1}{(3+x)(3-x)}$$
$$= \frac{a}{3+x} + \frac{b}{3-x}$$
$$= \frac{a(3-x)+b(3+x)}{(3+x)(3-x)}$$

So $1 = a(3-x) + b(3+x)$

Let $x = -3$, $1 = 6a$

$a = \frac{1}{6}$

Let $x = 3$, $1 = 6b$

$b = \frac{1}{6}$

So $\int_1^2 \frac{1}{9-x^2}\,dx$

$$= \int_1^2 \left(\frac{1}{6(3+x)} + \frac{1}{6(3-x)}\right)dx$$
$$= \left[\frac{1}{6}\log_e(3+x) - \frac{1}{6}\log_e(3-x)\right]_1^2$$
$$= \frac{1}{6}\left[\log_e(3+x) - \log_e(3-x)\right]_1^2$$
$$= \frac{1}{6}\left[\log_e\left(\frac{3+x}{3-x}\right)\right]_1^2$$
$$= \frac{1}{6}\left[\log_e\left(\frac{5}{-1}\right) - \log_e\left(\frac{4}{-2}\right)\right]$$
$$= \frac{1}{6}\left[\log_e(-5) - \log_e(-2)\right]$$
$$= \frac{1}{6}\log_e\frac{5}{2}$$

b $\int_0^1 \frac{-1}{\sqrt{4-x^2}}\,dx$

$$= \left[\cos^{-1}\frac{x}{2}\right]_0^1$$
$$= \cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}(0)$$
$$= \frac{\pi}{3} - \frac{\pi}{2}$$
$$= \frac{2\pi - 3\pi}{6}$$
$$= \frac{-\pi}{6}$$

d $\int_{-1}^2 \frac{dx}{1+x^2}$

$$= \left[\tan^{-1}x\right]_{-1}^2$$
$$= \tan^{-1}(2) - \tan^{-1}(-1)$$
$$\approx 1.893$$

e $\int_1^4 \frac{2}{x}\,dx$

$$= \left[2\log_e(x)\right]_1^4$$

$= 2\log_e(4) - 2\log_e(1)$
$= 2\log_e(4)$

f $\int_1^2 \frac{dx}{x(x+1)}$

$\frac{1}{x(x+1)} = \frac{a}{x} + \frac{b}{x+1}$

$= \frac{a(x+1)+bx}{x(x+1)}$

So $1 = a(x+1) + bx$
Let $x = -1$, $1 = -b$
$b = -1$
Let $x = 0$, $1 = a$
$a = 1$

Therefore $\int_1^2 \frac{dx}{x(x+1)}$

$= \int_1^2 \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

$= \left[\log_e(x) - \log_e(x+1)\right]_1^2$

$= \left[\log_e\left(\frac{x}{x+1}\right)\right]_1^2$

$= \log_e\left(\frac{2}{3}\right) - \log_e\left(\frac{1}{2}\right)$

$= \log_e\left(\frac{2}{3} \times \frac{2}{1}\right)$

$= \log_e\left(\frac{4}{3}\right)$

g $\int_{-1}^1 \frac{4x+10}{x^2+5x+6}\, dx$

$= \frac{4x+10}{x^2+5x+6} = \frac{4x+10}{(x+2)(x+3)}$

$= \frac{a}{x+2} + \frac{b}{x+3}$

$= \frac{a(x+3)+b(x+2)}{(x+2)(x+3)}$

So $4x + 10 = a(x+3) + b(x+2)$
Let $x = -3$, $-12 + 10 = -b$
$-2 = -b$
$b = 2$
Let $x = -2$, $-8 + 10 = a$
$a = 2$

So $\int_{-1}^1 \frac{4x+10}{x^2+5x+6} dx$

$= \int_{-1}^1 \left(\frac{2}{x+2} + \frac{2}{x+3}\right) dx$

$= \left[2\log_e(x+2) + 2\log_e(x+3)\right]_{-1}^1$

$= 2\left[\log_e[(x+2)(x+3)]\right]_{-1}^1$

$= 2\left[\log_e(3\times4) - \log_e(1\times2)\right]$

$= 2\log_e\left(\frac{12}{2}\right)$

$= 2\log_e(6)$

j $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{4x^2+9}$

$= \int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{4\left(\frac{4x^2+9}{4}\right)}$

$= \frac{1}{4}\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{x^2+\frac{9}{4}}$

$= \frac{1}{4} \times \frac{2}{3} \int_0^{\frac{\sqrt{3}}{2}} \frac{\frac{3}{2}}{x^2+\frac{9}{4}} dx$

$= \frac{1}{6}\int_0^{\frac{\sqrt{3}}{2}} \frac{\frac{3}{2}}{x^2+\frac{9}{4}} dx$

$= \frac{1}{6}\left[\tan^{-1}\left(\frac{x}{\frac{3}{2}}\right)\right]_0^{\frac{\sqrt{3}}{2}}$

$= \frac{1}{6}\left[\tan^{-1}\left(\frac{2x}{3}\right)\right]_0^{\frac{\sqrt{3}}{2}}$

$= \frac{1}{6}\left[\tan^{-1}\left(\frac{2}{3} \times \frac{\sqrt{3}}{3}\right) - \tan^{-1}(0)\right]$

$= \frac{1}{6}\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

$= \frac{1}{6}\tan^{-1}\frac{1}{\sqrt{3}}$

$= \frac{1}{6} \times \frac{\pi}{6}$

$= \frac{\pi}{36}$

l $\int_{\frac{1}{2}}^2 (2x-1)^{\frac{3}{2}}\, dx$

$= \left[\frac{1}{2} \times \frac{2}{5}(2x-1)^{\frac{5}{2}}\right]_{\frac{1}{2}}^2$

$= \frac{1}{5}\left[(2x-1)^{\frac{5}{2}}\right]_{\frac{1}{2}}^2$

$= \frac{1}{5}\left[(2\times2-1)^{\frac{5}{2}} - (2\times\frac{1}{2}-1)^{\frac{5}{2}}\right]$

$= \frac{1}{5}\,(3^{\frac{5}{2}} - 0)$

$= \frac{\sqrt{3^5}}{5}$

$= \frac{3^2\sqrt{3}}{5}$

$= \frac{9\sqrt{3}}{5}$

n $\int_0^3 (e^x + e^{-x})^2\, dx$

$= \int_0^3 (e^{2x} + 2e^x e^{-x} + e^{-2x})\, dx$

$= \int_0^3 (e^{2x} + 2 + e^{-2x})\, dx$

$= \left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right]_0^3$

$= \left[2x + \frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x}\right]_0^3$

$= \left[6+\frac{1}{2}e^{6}-\frac{1}{2}e^{-6}\right]-\left[0+\frac{1}{2}e^{0}-\frac{1}{2}e^{0}\right]$

$= 6+\frac{1}{2}e^{6}-\frac{1}{2}e^{-6}$

3 a $\int_0^2 2x^2\sqrt{x^3+1}\,dx$

Let $u = x^3+1$

$\frac{du}{dx} = 3x^2$ or $dx = \frac{du}{3x^2}$

When $x = 0$, $u = 0^3 + 1 = 1$

When $x = 2$, $u = 2^3 + 1 = 9$

So $\int_0^2 2x^2\sqrt{x^3+1}\,dx$

$= \int_1^9 2x^2\sqrt{u}\,\frac{du}{3x^2}$

$= \int_1^9 \frac{2\sqrt{u}}{3}du$

$\therefore$ **C**

b $\int_1^9 \frac{2\sqrt{u}}{3}du$

$= \frac{2}{3}\int_1^9 u^{\frac{1}{2}}\,du$

$= \frac{2}{3}\left[\frac{2}{3}u^{\frac{5}{2}}\right]_1^9$

$= \frac{4}{9}\left[9^{\frac{3}{2}}-1^{\frac{3}{2}}\right]$

$= \frac{4}{9}\left[3^3-1^3\right]$

$= \frac{4}{9}[27-1]$

$= \frac{4}{9}\times 26$

$= \frac{104}{9}$

$= 11\frac{5}{9}$

$\therefore$ **A**

4 a E

b $\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sqrt{1+\sin(x)}}dx$

Let $u = 1 + \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

When $x = 0$, $u = 1 + \sin(0) = 1$

When $x = \frac{\pi}{2}$, $u = 1 + \sin\left(\frac{\pi}{2}\right) = 2$

So $\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sqrt{1+\sin(x)}}dx$

$=.\int_1^2 \frac{\cos(x)}{\sqrt{u}}\times\frac{du}{\cos(x)}$

$= \int_1^2 \frac{1}{\sqrt{u}}\,du$

$= \int_1^2 u^{\frac{-1}{2}}\,du$

$\therefore$ **D**

c $\int_1^2 u^{\frac{-1}{2}}\,du$

$= \left[2u^{\frac{1}{2}}\right]_1^2$

$= 2\times 2^{\frac{1}{2}} - 2\times 1^{\frac{1}{2}}$

$= 2\sqrt{2}-2$

$\therefore$ **B**

5 a $\int_0^2 x^2(2+x^3)^3\,dx$

Let $u = 2+x^3$

$\frac{du}{dx} = 3x^2$ or $dx = \frac{du}{3x^2}$

When $x = 0$, $u = 2 + 0^3 = 2$

When $x = 2$, $u = 2 + 2^3 = 10$

So $\int_0^2 x^2(2+x^3)^3\,dx$

$= \int_2^{10} x^2u^3\frac{du}{3x^2}$

$= \int_2^{10} \frac{u^3}{3}\,dx$

b $\int_1^{\sqrt{2}} \frac{4x}{(x^2-3)^2}dx$

Let $u = x^2 - 3$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

When $x = 1$, $u = 1^2 - 3 = -2$

When $x = \sqrt{2}$, $u = (\sqrt{2})^2 - 3 = 2 - 3 = -1$

So $\int_1^{\sqrt{2}} \frac{4x}{(x^2-3)^2}dx$

$= \int_{-2}^{-1} \frac{4x}{u^2}\times\frac{du}{2x}$

$= \int_{-2}^{-1} \frac{2}{u^2}\,du$

$= \int_{-2}^{-1} 2u^{-2}\,du$

c $\int_0^1 x\sqrt{x^2+1}\,dx$

Let $u = x^2 + 1$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

When $x = 0$, $u = 0^2 + 1 = 1$

When $x = 1$, $u = 1^2 + 1 = 2$

So $\int_0^1 x\sqrt{x^2+1}\,dx$

$= \int_1^2 x\sqrt{u}\,\frac{du}{2x}$

$= \int_1^2 \frac{u^{\frac{1}{2}}}{2}du$

d $\int_2^4 (x-1)\sqrt{x^2-2x}\,dx$

Let $u = x^2 – 2x$

$\frac{du}{dx} = 2x - 2$ or $dx = \frac{du}{2x-2}$

When $x = 2$, $u = 2^2 - 2 \times 2$
$= 0$

When $x = 4$, $u = 4^2 - 2 \times 4$
$= 8$

So $\int_2^4 (x-1)\sqrt{x^2+2x}\,dx$

$= \int_0^8 (x-1)\sqrt{u}\,\frac{du}{2x-2}$

$= \int_0^8 (x-1)u^{\frac{1}{2}}\frac{du}{2(x-1)}$

$= \int_0^8 \frac{u^{\frac{1}{2}}}{2}\,du$

e $\int_1^2 x\sqrt{x-1}\,dx$

Let $u = x - 1$ and $x = u + 1$

$\frac{du}{dx} = 1$ or $dx = du$

When $x = 1$, $u = 1 - 1$
$= 0$

When $x = 2$, $u = 2 - 1$
$= 1$

So $\int_1^2 x\sqrt{x-1}\,dx$

$= \int_0^1 (u+1)\sqrt{u}\,du$

$= \int_0^1 (u+1)u^{\frac{1}{2}}\,du$

$= \int_0^1 (u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du$

f $\int_0^3 \frac{x^2}{\sqrt{x+1}}\,dx$

Let $u = x + 1$ and $x = u - 1$

$\frac{du}{dx} = 1$ or $dx = du$

When $x = 0$, $u = 0 + 1$
$= 1$

When $x = 3$, $u = 3 + 1$
$= 4$

So $\int_0^3 \frac{x^2}{\sqrt{x+1}}\,dx$

$= \int_1^4 \frac{(u-1)^2}{\sqrt{u}}\,du$

$= \int_1^4 \frac{u^2-2u+1}{u^{\frac{1}{2}}}\,du$

$= \int_1^4 \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + u^{\frac{-1}{2}}\right)du$

g $\int_1^3 \frac{\log_e(x)}{x}dx$

Let $u = \log_e(x)$

$\frac{du}{dx} = \frac{1}{x}$ or $dx = x\,du$

When $x = 1$, $u = \log_e(1)$
$= 0$

When $x = 3$, $u = \log_e(3)$

So $\int_1^3 \frac{\log_e(x)}{x}\,dx$

$= \int_0^{\log_e(3)} \frac{u}{x}x\,du$

$= \int_0^{\log_e(3)} u\,du$

h $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(x)e^{\cos(x)}\,dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

When $x = \frac{\pi}{3}$, $u = \cos\left(\frac{\pi}{3}\right)$
$= \frac{1}{2}$

When $x = \frac{\pi}{2}$, $u = \cos\left(\frac{\pi}{2}\right)$
$= 0$

So $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(x)e^{\cos(x)}\,dx$

$= \int_{\frac{1}{2}}^{0} \sin(x)e^u\frac{du}{-\sin(x)}$

$= \int_{\frac{1}{2}}^{0} -e^u\,du$

i $\int_0^1 x(1-x)^{10}\,dx$

Let $u = 1 - x$ and $x = 1 - u$

$\frac{du}{dx} = -1$ or $dx = -du$

When $x = 0$, $u = 1 - 0$
$= 1$

When $x = 1$, $u = 1 - 1$
$= 0$

So $\int_0^1 x(1-x)^{10}\,dx$

$= \int_0^1 (1-u)(u)^{10} \times -du$

$= \int_0^1 (u-1)u^{10}\,du$

$= \int_0^1 (u^{11} - u^{10})\,du$

j $\int_0^{\frac{\pi}{2}} \cos(x)\sqrt{\sin(x)}\,dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

When $x = 0$, $u = \sin(0)$
$= 0$

When $x = \frac{\pi}{2}$, $u = \sin\left(\frac{\pi}{2}\right)$
$= 1$

So $\int_0^{\frac{\pi}{2}} \cos(x)\sqrt{\sin(x)}\ dx$

$= \int_0^1 \cos(x)\sqrt{u}\ \frac{du}{\cos(x)}$

$= \int_0^1 u^{\frac{1}{2}}\ du$

k $\int_0^{\frac{\pi}{4}} \tan^3(x)\sec^2(x)\,dx$

Let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$ or $dx = \frac{du}{\sec^2(x)}$

When $x = 0$, $u = \tan(0) = 0$

When $x = \frac{\pi}{4}$, $u = \tan\left(\frac{\pi}{4}\right) = 1$

So $\int_0^{\frac{\pi}{4}} \tan^3(x)\sec^2(x)\,dx$

$= \int_0^1 u^3 \sec^2(x)\frac{du}{\sec^2(x)}$

$= \int_0^1 u^3\,du$

l $\int_0^{\frac{\pi}{2}} x\sin(x^2)\,dx$

Let $u = x^2$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

When $x = 0$, $u = 0^2 = 0$

When $x = \frac{\pi}{2}$, $u = \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}$

So $\int_0^{\frac{\pi}{2}} x\sin(x^2)\,dx$

$= \int_0^{\frac{\pi^2}{4}} x\sin(u)\frac{du}{dx}$

$= \int_0^{\frac{\pi^2}{4}} \frac{\sin(u)}{2}\,du$

m $\int_{\frac{\pi}{2}}^{\pi} \cos^3(x)\,dx$

$= \int_{\frac{\pi}{2}}^{\pi} \cos^2(x)\cos(x)\,dx$

$= \int_{\frac{\pi}{2}}^{\pi} (1-\sin^2(x))\cos(x)\,dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

When $x = \frac{\pi}{2}$, $u = \sin\left(\frac{\pi}{2}\right) = 1$

When $x = \pi$, $u = \sin(\pi) = 0$

So $\int_{\frac{\pi}{2}}^{\pi} (1-\sin^2(x))\cos(x)\,dx$

$= \int_1^0 (1-u^2)\cos(x)\frac{du}{\cos(x)}$

$= \int_1^0 (1-u^2)\ du$

n $\int_0^1 \frac{e^x}{\sqrt{e^x+1}}\ dx$

Let $u = e^x + 1$

$\frac{du}{dx} = e^x$ or $dx = \frac{du}{e^x}$

When $x = 0$, $u = e^0 + 1 = 1 + 1 = 2$

When $x = 1$, $u = e^1 + 1 = e + 1$

So $\int_0^1 \frac{e^x}{\sqrt{e^x+1}}\ dx$

$= \int_2^{e+1} \frac{e^x}{\sqrt{u}} \times \frac{du}{e^x}$

$= \int_2^{e+1} \frac{1}{u^{\frac{1}{2}}}\ du$

$= \int_2^{e+1} u^{\frac{-1}{2}}\ du$

6 a $\int_0^2 x^2(2+x^3)^3\,dx$

$= \int_2^{10} \frac{u^3}{3}\ du$

$= \frac{1}{3}\int_2^{10} u^3\ du$

$= \frac{1}{3}\left[\frac{1}{4}u^4\right]_2^{10}$

$= \frac{1}{12}\left[10^4 - 2^4\right]$

$= \frac{1}{12}[10000-16]$

$= 832$

b $\int_1^{\sqrt{2}} \frac{4x}{(x^2-3)^2}\ dx$

$= \int_{-2}^{-1} 2u^{-2}\ du$

$= 2\int_{-2}^{-1} u^{-2}\ du$

$= 2\left[-u^{-1}\right]_{-2}^{-1}$

$= -2\left[(-1)^{-1} - (-2)^{-1}\right]$

$= -2\left[-1 + \frac{1}{2}\right]$

$= -2 \times \frac{-1}{2}$

$= 1$

c $\int_0^1 x\sqrt{x^2+1}\ dx$

$= \int_1^2 \frac{u^{\frac{1}{2}}}{2}\ du$

$= \frac{1}{2}\int_1^2 u^{\frac{1}{2}}\ du$

$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^2$

$= \frac{1}{3}\left[2^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$

$= \frac{1}{3}\left(\sqrt{2^3} - 1\right)$

$= \frac{2\sqrt{2}-1}{3}$

d $\int_2^4 (x-1)\sqrt{x^2-2x}\ dx$

$= \int_0^8 \frac{u^{\frac{1}{2}}}{2}\ du$

$= \frac{1}{2}\int_0^8 u^{\frac{1}{2}}\ du$

$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^8$

$= \frac{1}{3}\left[8^{\frac{3}{2}} - 0^{\frac{3}{2}}\right]$

$= \frac{\sqrt{8^3}}{3}$

$= \frac{\sqrt{2^9}}{3}$

$= \frac{2^4\sqrt{2}}{3}$

$= \frac{16\sqrt{2}}{3}$

e $\int_1^2 x\sqrt{x-1}\, dx$

$= \int_0^1 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

$= \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]_0^1$

$= \left[\frac{2}{5}(1)^{\frac{5}{2}} + \frac{2}{3}(1)^{\frac{3}{2}}\right] - \left[\frac{2}{5}(0)^{\frac{5}{2}} + \frac{2}{3}(0)^{\frac{3}{2}}\right]$

$= \frac{2}{5} + \frac{2}{3}$

$= \frac{6+10}{15}$

$= \frac{16}{15}$

$= 1\frac{1}{15}$

f $\int_1^4 \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + u^{\frac{-1}{2}}\right) du$

$= \left[\frac{2}{5}u^{\frac{5}{2}} - 2\times\frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right]_1^4$

$= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right]_1^4$

$= \left[\frac{2}{5}(4)^{\frac{5}{2}} - \frac{4}{3}(4)^{\frac{3}{2}} + 2(4)^{\frac{1}{2}}\right] - \left[\frac{2}{5}(1)^{\frac{5}{2}} - \frac{4}{3}(1)^{\frac{3}{2}} + 2(1)^{\frac{1}{2}}\right]$

$= \left[\frac{2}{5}\times 2^5 - \frac{4}{3}\times 2^3 + 2\times 2\right] - \left[\frac{2}{5} - \frac{4}{3} + 2\right]$

$= \frac{64}{5} - \frac{32}{3} + 4 - \frac{2}{5} + \frac{4}{3} - 2$

$= \frac{62}{5} - \frac{28}{3} + 2$

$= \frac{62\times 3 - 28\times 5 + 2\times 15}{15}$

$= \frac{186-140+30}{15}$

$= \frac{76}{15}$

$= 5\frac{1}{15}$

g $\int_1^3 \frac{\log_e(x)}{x}\ dx$

$= \int_0^{\log_e(3)} u\ du$

$= \left[\frac{1}{2}u^2\right]_0^{\log_e(3)}$

$= \frac{1}{2}(\log_e(3))^2 - \frac{1}{2}(0)^2$

$= \frac{(\log_e(3))^2}{2}$

h $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(x)e^{\cos(x)}\ dx$

$= \int_{\frac{1}{2}}^{0} -e^u\ du$

$= \left[-e^u\right]_{\frac{1}{2}}^{0}$

$= -e^0 + e^{\frac{1}{2}}$

$= 1 + e^{\frac{1}{2}}$

$= e^{\frac{1}{2}} - 1$

i $\int_0^1 x(1-x)^{10}\, dx$

$= \int_1^0 (u^{11} - u^{10})\ du$

$= \left[\frac{1}{12}u^{12} - \frac{1}{11}u^{11}\right]_1^0$

$= \left[\frac{1}{12}(0)^{12} - \frac{1}{11}(0)^{11}\right] - \left[\frac{1}{12}(1)^{12} - \frac{1}{11}(1)^{11}\right]$

$= -\left[\frac{1}{12} - \frac{1}{11}\right]$

$= \frac{1}{11} - \frac{1}{12}$

$= \frac{12-11}{132}$

$= \frac{1}{132}$

j $\int_0^{\frac{\pi}{2}} \cos(x)\sqrt{\sin(x)}\ dx$

$= \int_0^1 u^{\frac{1}{2}}\, du$

$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^1$

$= \frac{2}{3}(1)^{\frac{3}{2}} - \frac{2}{3}(0)^{\frac{3}{2}}$

$= \frac{2}{3}$

k $\int_0^{\frac{\pi}{4}} \tan^3(x)\sec^2(x)\,dx$

$= \int_0^1 u^3\,du$

$= \left[\frac{1}{4}u^4\right]_0^1$

$= \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4$

$= \frac{1}{4}$

l $\int_0^{\frac{\pi}{2}} x\sin(x)^2\,dx$

$= \int_0^{\frac{\pi^2}{4}} \frac{\sin(u)}{2}du$

$= \frac{1}{2}\int_0^{\frac{\pi^2}{4}} \sin(u)\,du$

$= \frac{1}{2}[-\cos(u)]_0^{\frac{\pi^2}{4}}$

$= \frac{-1}{2}[\cos(u)]_0^{\frac{\pi^2}{4}}$

$= \frac{-1}{2}\left[\cos\left(\frac{\pi^2}{4}\right) - \cos(0)\right]$

$= \frac{-1}{2}\left[\cos\left(\frac{\pi^2}{4}\right) - 1\right]$

$= \frac{1-\cos\left(\frac{\pi^2}{4}\right)}{2}$

≈ 0.89

m $\int_{\frac{\pi}{2}}^{\pi} \cos^3(x)\,dx$

$= \int_1^0 (1-u^2)\,du$

$= \left[u - \frac{1}{3}u^3\right]_1^0$

$= \left[0 - \frac{1}{3}(0)^3\right] - \left[1 - \frac{1}{3}(1)^3\right]$

$= -\left[1 - \frac{1}{3}\right]$

$= \frac{-2}{3}$

n $\int_0^1 \frac{e^x}{\sqrt{e^x+1}}dx$

$\int_2^{e+1} u^{\frac{-1}{2}}\,du$

$= \left[2u^{\frac{1}{2}}\right]_2^{e+1}$

$= 2(e+1)^{\frac{1}{2}} - 2\times 2^{\frac{1}{2}}$

$= 2\sqrt{e+1} - 2\sqrt{2}$

≈ 1.028

7 a $\int_{-2}^0 4xe^{x^2}\,dx$

Let $u = x^2$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

When $x = -2$, $u = (-2)^2 = 4$

When $x = 0$, $u = 0^2 = 0$

So $\int_{-2}^0 4xe^{x^2}\,dx$

$= \int_4^0 4xe^u \frac{du}{2x}$

$= 2\int_4^0 e^u\,du$

$= 2\left[e^u\right]_4^0$

$= 2(e^0 - e^4)$

$= 2 - 2e^4$

b $\int_0^1 (4x+7)\sqrt{2x^2+7x}\,dx$

Let $u = 2x^2 + 7x$

$\frac{du}{dx} = 4x + 7$ or $dx = \frac{du}{4x+7}$

When $x = 0$, $u = 2(0)^2 + 7(0) = 0$

When $x = 1$, $u = 2(1)^2 + 7(1) = 2 + 7 = 9$

So $\int_0^1 (4x+7)\sqrt{2x^2+7x}\,dx$

$= \int_0^9 (4x+7)\sqrt{u}\frac{du}{4x+7}$

$= \int_0^9 u^{\frac{1}{2}}\,du$

$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^9$

$= \frac{2}{3}(9)^{\frac{3}{2}} - \frac{2}{3}(0)^{\frac{3}{2}}$

$= \frac{2}{3}\times 3^3$

$= 2 \times 3^2$

$= 18$

c $\int_{-3}^{-2} 2\sqrt{x+3}\,dx$

Let $u = x + 3$

$\frac{du}{dx} = 1$ or $dx = du$

When $x = -3$, $u = -3 + 3 = 0$

When $x = -2$, $u = -2 + 3 = 1$

So $\int_{-3}^{-2} 2\sqrt{x+3}\,dx$

$= \int_0^1 2\sqrt{u}\,du$

$= 2\int_0^1 u^{\frac{1}{2}}\,du$

$= 2\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^1$

$= \frac{4}{3}\left[(1)^{\frac{3}{2}} - (0)^{\frac{3}{2}}\right]$

$= \frac{4}{3}$

d $\int_{-1}^{0}\frac{x+1}{\sqrt{1-x}}\,dx$

Let $u = 1 - x$ and $x = 1 - u$

$\frac{du}{dx} = -1$ or $dx = -du$

When $x = -1$, $u = 1 - (-1)$
$= 1 + 1$
$= 2$

When $x = 0$, $u = 1 - 0$
$= 1$

So $\int_{-1}^{0}\frac{x+1}{\sqrt{1-x}}\,dx$

$= \int_2^1 \frac{(1-u)+1}{\sqrt{u}} \times -du$

$= -\int_2^1 \frac{2-u}{\sqrt{u}}\,du$

$= \int_2^1 \frac{u-2}{u^{\frac{1}{2}}}\,du$

$= \int_2^1 (u^{\frac{1}{2}} - 2u^{\frac{-1}{2}})\,du$

$= \left[\frac{2}{3}u^{\frac{3}{2}} - 2\times 2u^{\frac{1}{2}}\right]_2^1$

$= \left[\frac{2}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right]_2^1$

$= \left[\frac{2}{3}(1)^{\frac{3}{2}} - 4(1)^{\frac{1}{2}}\right] - \left[\frac{2}{3}(2)^{\frac{3}{2}} - 4(2)^{\frac{1}{2}}\right]$

$= \left[\frac{2}{3} - 4\right] - \left[\frac{2}{3}\sqrt{2^3} - 4\sqrt{2}\right]$

$= \frac{2-12}{3} - \left[\frac{4}{3}\sqrt{2} - 4\sqrt{2}\right]$

$= \frac{-10}{3} - \frac{(4\sqrt{2} - 12\sqrt{2})}{3}$

$= \frac{-10}{3} - \frac{(-8\sqrt{2})}{3}$

$= \frac{-10+8\sqrt{2}}{3}$

$= \frac{8\sqrt{2}-10}{3}$

e $\int_0^{\frac{\pi}{3}} \sin(x)\cos^4(x)\,dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

When $x = 0$, $u = \cos(0)$
$= 1$

When $x = \frac{\pi}{3}$, $u = \cos\left(\frac{\pi}{3}\right)$
$= \frac{1}{2}$

So $\int_0^{\frac{\pi}{3}} \sin(x)\cos^4(x)\,dx$

$= \int_1^{\frac{1}{2}} \sin(x)u^4 \frac{du}{-\sin(x)}$

$= -\int_1^{\frac{1}{2}} u^4\,du$

$= -\left[\frac{1}{5}u^5\right]_1^{\frac{1}{2}}$

$= \frac{-1}{5}\left[u^5\right]_1^{\frac{1}{2}}$

$= \frac{-1}{5}\left[\left(\frac{1}{2}\right)^5 - 1^5\right]$

$= \frac{-1}{5}\left[\frac{1}{32} - 1\right]$

$= \frac{1}{5}\left[1 - \frac{1}{32}\right]$

$= \frac{1}{5}\left[\frac{32-1}{32}\right]$

$= \frac{1}{5}\left[\frac{31}{32}\right]$

$= \frac{31}{160}$

f $\int_0^2 \frac{x+5}{x^2+4x+3}\,dx$

$\frac{x+5}{x^2+4x+3} = \frac{x+5}{(x+3)(x+1)}$

$= \frac{a}{x+3} + \frac{b}{x+1}$

$= \frac{a(x+1)+b(x+3)}{(x+3)(x+1)}$

So $x + 5 = a(x + 1) + b(x + 3)$

Let $x = -3$, $2 = -2a$
$a = -1$

Let $x = -1$, $4 = 2b$
$b = 2$

So $\int_0^2 \frac{x+5}{x^2+4x+3}\,dx$

$= \int_0^2 \left(\frac{-1}{x+3} + \frac{2}{x+1}\right)dx$

$= \int_0^2 \left(\frac{2}{x+1} - \frac{1}{x+3}\right)dx$

$= \left[2\log_e(x+1) - \log_e(x+3)\right]_0^2$

$= \left[\log_e\left[(x+1)^2\right] - \log_e(x+3)\right]_0^2$

$= \left[\log_e\left[\frac{(x+1)^2}{x+3}\right]\right]_0^2$

$= \log_e\left(\frac{3^2}{5}\right) - \log_e\left(\frac{1^2}{3}\right)$

$= \log_e\left(\frac{9}{5}\right) - \log e\left(\frac{1}{3}\right)$

$= \log_e\left(\frac{9}{5}\times\frac{3}{1}\right)$

$= \log_e\left(\frac{27}{5}\right)$

g $\int_{-1}^{1}\frac{1}{\sqrt{4-x^2}}dx$

$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-1}^{1}$

$= \sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(\frac{-1}{2}\right)$

$= \frac{\pi}{6}-\left(\frac{-\pi}{6}\right)$

$= \frac{\pi}{3}$

h $\int_{0}^{3}\frac{1}{(x-3)^2+1}dx$

Let $u = x - 3$

$\frac{du}{dx} = 1$ or $dx = du$

When $x = 0$, $u = 0 - 3$
$= -3$

When $x = 3$, $u = 3 - 3$
$= 0$

So $\int_{0}^{3}\frac{1}{(x-3)^2+1}dx$

$= \int_{-3}^{0}\frac{1}{u^2+1}du$

$= \left[\tan^{-1}u\right]_{-3}^{0}$

$= \tan^{-1}(0) - \tan^{-1}(-3)$

$= -\tan^{-1}(-3)$

$= \tan^{-1}(3)$

≈ 1.249

i $\int_{-1}^{1}\frac{-1}{\sqrt{4-(x-1)^2}}\,dx$

Let $u = x - 1$

$\frac{du}{dx} = 1$ or $dx = du$

When $x = -1$, $u = -1 - 1$
$= -2$

When $x = 1$, $u = 1 - 1$
$= 0$

So $\int_{-1}^{1}\frac{-1}{\sqrt{4-(x-1)^2}}\,dx$

$= \int_{-2}^{0}\frac{-1}{\sqrt{4-u^2}}du$

$= \left[\cos^{-1}\left(\frac{u}{2}\right)\right]_{-2}^{0}$

$= \cos^{-1}(0) - \cos^{-1}\left(\frac{-2}{2}\right)$

$= \cos^{-1}(0) - \cos^{-1}(-1)$

$= \frac{\pi}{2}-\pi$

$= \frac{-\pi}{2}$

j $\int_{0}^{\pi}\sin^3(x)\cos^2(x)dx$

$= \int_{0}^{\pi}\sin(x)(1-\cos^2(x))\cos^2(x)dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

When $x = 0$, $u = \cos(0)$
$= 1$

When $x = \pi$, $u = \cos(\pi)$
$= -1$

So $\int_{0}^{\pi}\sin(x)(1-\cos^2(x))\cos^2(x)dx$

$= \int_{1}^{-1}\sin(x)(1-u^2)u^2\frac{du}{-\sin(x)}$

$= \int_{1}^{-1}(u^2-1)u^2\,du$

$= \int_{1}^{-1}(u^4-u^2)du$

$= \left[\frac{1}{5}u^5-\frac{1}{3}u^3\right]_{1}^{-1}$

$= \left[\frac{1}{5}(-1)^5-\frac{1}{3}(-1)^3\right]-\left[\frac{1}{5}(1)^5-\frac{1}{3}(1)^3\right]$

$= \frac{-1}{5}+\frac{1}{3}-\frac{1}{5}+\frac{1}{3}$

$= \frac{2}{3}-\frac{2}{5}$

$= \frac{10-6}{15}$

$= \frac{4}{15}$

k $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cot(x)\,dx$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos(x)}{\sin(x)}dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

When $x = \frac{\pi}{4}$, $u = \sin\left(\frac{\pi}{4}\right)$
$= \frac{1}{\sqrt{2}}$

When $x = \frac{\pi}{2}$, $u = \sin\left(\frac{\pi}{2}\right)$
$= 1$

So $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos(x)}{\sin(x)}dx$

$= \int_{\frac{1}{\sqrt{2}}}^{1}\frac{\cos(x)}{u}\times\frac{du}{\cos(x)}$

$= \int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{u}du$

$= \left[\log_e(u)\right]_{\frac{1}{\sqrt{2}}}^{1}$

$= \log_e(1) - \log_e\left(\frac{1}{\sqrt{2}}\right)$

$= -\log_e\left(\frac{1}{\sqrt{2}}\right)$

$= -\log_e\left(\frac{\sqrt{2}}{2}\right)$

l $\int_5^6 \frac{3x-10}{x^2-7x+12}dx$

$\frac{3x-10}{x^2-7x+12} = \frac{3x-10}{(x-3)(x-4)}$

$= \frac{a}{x-3} + \frac{b}{x-4}$

$= \frac{a(x-4)+b(x-3)}{(x-3)(x-4)}$

So $3x - 10 = a(x-4) + b(x-3)$

Let $x = 3$, $9 - 10 = -a$

$-1 = -a$

$a = 1$

Let $x = 4$, $12 - 10 = b$

$b = 2$

So $\int_5^6 \frac{3x-10}{x^2-7x+12}dx$

$= \int_5^6 \left(\frac{1}{x-3} + \frac{2}{x-4}\right)dx$

$= [\log_e(x-3) + 2\log_e(x-4)]_5^6$

$= \left[\log_e(x-3) + \log_e[(x-4)^2]\right]_5^6$

$= \left[\log_e[(x-3)(x-4)^2]\right]_5^6$

$= \log_e[3(2)^2] - \log_e[2(1)^2]$

$= \log_e(12) - \log_e(2)$

$= \log_e\left(\frac{12}{2}\right)$

$= \log_e(6)$

≈ 1.79

m $\int_{-1}^1 \frac{2x^2}{x^2+1}dx$

$= \frac{2x^2}{x^2+1} = \frac{2x^2+2-2}{x^2+1}$

$= \frac{2(x^2+1)}{x^2+1} - \frac{2}{x^2+1}$

$= 2 - \frac{2}{x^2+1}$

So $\int_{-1}^1 \frac{2x^2}{x^2+1}dx$

$= \int_{-1}^1 \left(2 - \frac{2}{x^2+1}\right)dx$

$= \left[2x - 2\tan^{-1}(x)\right]_{-1}^1$

$= \left[2\times 1 - 2\tan^{-1}(1)\right] - \left[2\times(-1) - 2\tan^{-1}(-1)\right]$

$= 2 - \frac{2\pi}{4} - \left[-2 - 2\left(\frac{-\pi}{4}\right)\right]$

$= 2 - \frac{\pi}{2} + 2 - \frac{\pi}{2}$

$= 4 - \pi$

n $\int_0^{\frac{\pi}{2}} 2\sin(2x)\cos(x)\,dx$

$= \int_0^{\frac{\pi}{2}} 2(2\sin(x)\cos(x))\cos(x)dx$

$= 4\int_0^{\frac{\pi}{2}} \sin(x)\cos^2(x)\,dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

When $x = 0$, $u = \cos(0)$

$= 1$

When $x = \frac{\pi}{2}$, $u = \cos\left(\frac{\pi}{2}\right)$

$= 0$

So $4\int_0^{\frac{\pi}{2}} \sin(x)\cos^2(x)\,dx$

$= 4\int_1^0 \sin(x)u^2 \frac{du}{-\sin(x)}$

$= -4\int_1^0 (u^2)\,du$

$= -4\left[\frac{1}{3}u^3\right]_1^0$

$= \frac{-4}{3}\left[u^3\right]_1^0$

$= \frac{-4}{3}[0^3 - 1^3]$

$= \frac{4}{3}$

o $\int_0^1 (2x+1)e^{x^2+x}\,dx$

Let $u = x^2 + x$

$\frac{du}{dx} = 2x+1$ or $dx = \frac{du}{2x+1}$

When $x = 0$, $u = 0^2 + 0$

$= 0$

When $x = 1$, $u = 1^2 + 1$

$= 2$

So $\int_0^1 (2x+1)e^{x^2+x}\,dx$

$= \int_0^2 (2x+1)e^u \frac{du}{2x+1}$

$= \int_0^2 e^u\,du$

$= \left[e^u\right]_0^2$

$= e^2 - e^0$

$= e^2 - 1$

p $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (2 + \tan^2(x))\,dx$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1 + (-1 + \tan^2(x))\,dx$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1 + \sec^2(x))\,dx$

$= [x + \tan(x)]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \left[\frac{\pi}{3} + \tan\left(\frac{\pi}{3}\right)\right] - \left[\frac{\pi}{4} + \tan\left(\frac{\pi}{4}\right)\right]$

$= \frac{\pi}{3} + \sqrt{3} - \frac{\pi}{4} - 1$

$= \frac{4\pi - 3\pi}{12} + \sqrt{3} - 1$

$= \frac{\pi}{12} + \sqrt{3} - 1$

q $\int_2^5 \frac{x^2}{\sqrt{x-1}}dx$

Let $u = x - 1$ and $x = u + 1$

$\frac{du}{dx} = 1$ or $dx = du$

When $x = 2$, $u = 2 - 1 = 1$

When $x = 5$, $u = 5 - 1 = 4$

So $\int_2^5 \frac{x^2}{\sqrt{u-1}}dx$

$= \int_1^4 \frac{(u+1)^2}{\sqrt{u}}du$

$= \int_1^4 \frac{u^2+2u+1}{u^{\frac{1}{2}}}du$

$= \int_1^4 (u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + u^{\frac{-1}{2}})du$

$= \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right]_1^4$

$= \left[\frac{2}{5}(4)^{\frac{5}{2}} + \frac{4}{3}(4)^{\frac{3}{2}} + 2(4)^{\frac{1}{2}}\right] - \left[\frac{2}{5}(1)^{\frac{5}{2}} + \frac{4}{3}(1)^{\frac{3}{2}} + 2(1)^{\frac{1}{2}}\right]$

$= \left[\frac{2}{5}(2^5) + \frac{4}{3}(2^3) + 2(2)\right] - \left[\frac{2}{5} + \frac{4}{3} + 2\right]$

$= \frac{64}{5} + \frac{32}{3} + 4 - \frac{2}{5} - \frac{4}{3} - 2$

$= \frac{62}{5} + \frac{28}{3} + 2$

$= \frac{62\times3 + 28\times5 + 2\times15}{15}$

$= \frac{186+140+30}{15}$

$= \frac{356}{15}$

$= 23\frac{11}{15}$

r $\int_3^4 \frac{2x^3+x^2-2x-4}{x^2-4}dx$

$\frac{2x^3+x^2-2x-4}{x^2-4} = \frac{2x^3-8x+x^2+6x-4}{x^2-4}$

$= \frac{2x(x^2-4)}{x^2-4} + \frac{x^2+6x-4}{x^2-4}$

$= 2x + \frac{x^2-4+6x}{x^2-4}$

$= 2x + \frac{x^2-4}{x^2-4} + \frac{6x}{x^2-4}$

$= 2x + 1 + \frac{6x}{x^2-4}$

So $\int_3^4 \frac{2x^3+x^2-2x-4}{x^2-4}dx$

$= \int_3^4 \left(2x+1+\frac{6x}{x^2-4}\right)dx$

$= \int_3^4 (2x+1)dx + 6\int_3^4 \frac{x}{x^2-4}dx$

Let $u = x^2 - 4$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

When $x = 3$, $u = 3^2 - 4 = 9 - 4 = 5$

When $x = 4$, $u = 4^2 - 4 = 16 - 4 = 12$

So $\int_3^4 (2x+1)dx + 6\int_3^4 \frac{x}{x^2-4}dx$

$= \int_3^4 (2x+1)dx + 6\int_5^{12} \frac{x}{u}\times\frac{du}{2x}$

$= \int_3^4 (2x+1)dx + 3\int_5^{12} \frac{1}{u}du$

$= \left[x^2+x\right]_3^4 + 3\left[\log_e(u)\right]_5^{12}$

$= \left[4^2+4\right] - \left[3^2+3\right] + 3\left[\log_e(12) - \log_e(5)\right]$

$= 16 + 4 - [9+3] + 3\log_e\left(\frac{12}{5}\right)$

$= 20 - 12 + 3\log_e\left(\frac{12}{5}\right)$

$= 8 + 3\log_e\frac{12}{5}$

≈ 10.626

8 $\int_0^1 \sqrt{1-x^2}\, dx$

Let $x = \sin(\theta)$

$\frac{dx}{d\theta} = \cos(\theta)$ or $dx = \cos(\theta)\, d\theta$

When $x = 0$, $0 = \sin(\theta)$, $\theta = 0$

When $x = 1$, $1 = \sin(\theta)$, $\theta = \frac{\pi}{2}$

So $\int_0^1 \sqrt{1-x^2}\, dx$

$= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2(\theta)}\cos(\theta)d\theta$

$= \int_0^{\frac{\pi}{2}} \cos(\theta)\cos(\theta)d\theta$

$= \int_0^{\frac{\pi}{2}} \cos^2(\theta)d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos(2\theta))\, d\theta$

$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin(\pi)\right) - \left(0 + \frac{1}{2}\sin(0)\right)\right]$

$= \frac{1}{2}\left(\frac{\pi}{2}\right)$

$= \frac{\pi}{4}$

9 $\int_0^{\sqrt{3}} \sqrt{4-x^2}\, dx$

Let $x = 2\sin(\theta)$

$\frac{dx}{d\theta} = 2\cos(\theta)$ or $dx = 2\cos(\theta)\, d\theta$

When $x = 0$, $0 = 2\sin(\theta)$, $\theta = 0$

When $x = \sqrt{3}$, $\quad \sqrt{3} = 2\sin(\theta)$

$$\sin(\theta) = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{3}$$

So $\int_0^{\sqrt{3}} \sqrt{4-x^2}\,dx$

$= \int_0^{\frac{\pi}{3}} \sqrt{4-(2\sin(\theta))^2} \times 2\cos(\theta)\,d\theta$

$= 2\int_0^{\frac{\pi}{3}} \sqrt{4(1-\sin^2(\theta))}\cos(\theta)\,d\theta$

$= 2\int_0^{\frac{\pi}{3}} 2\cos(\theta)\cos(\theta)\,d\theta$

$= 4\int_0^{\frac{\pi}{3}} \cos^2(\theta)\,d\theta$

$= \frac{4}{2}\int_0^{\frac{\pi}{3}} (1+\cos(2\theta)\,d\theta$

$= 2\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{3}}$

$= 2\left[\left(\frac{\pi}{3} + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)\right) - \left(0 + \frac{1}{2}\sin(0)\right)\right]$

$= 2\left(\frac{\pi}{3} + \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right)$

$= 2\left(\frac{\pi}{3} + \frac{1}{2} \times \frac{\sqrt{3}}{2}\right)$

$= \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$

10 $\int_0^1 \frac{dx}{(1+x^2)^2}$

Let $x = \tan(\theta)$

$\frac{dx}{d\theta} = \sec^2(\theta)$ or $dx = \sec^2(\theta)\,d\theta$

When $x = 0$, $\quad 0 = \tan(\theta)$

$\theta = \tan^{-1}(0)$

$= 0$

When $x = 1$, $\quad 1 = \tan(\theta)$

$\theta = \tan^{-1}(1)$

$= \frac{\pi}{4}$

So $\int_0^1 \frac{dx}{(1+x^2)^2}$

$= \int_0^{\frac{\pi}{4}} \frac{1}{(1+\tan^2(\theta))^2}\sec^2(\theta)\,d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{\sec^2(\theta)}{(\sec^2(\theta))^2}\,d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{\sec^2(\theta)}{\sec^4(\theta)}\,d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2(\theta)}\,d\theta$

$= \int_0^{\frac{\pi}{4}} \cos^2(\theta)\,d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos(2\theta))\,d\theta$

$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{4} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right) - \left(0 + \frac{1}{2}\sin(0)\right)\right]$

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2} \times 1\right)$

$= \frac{\pi}{8} + \frac{1}{4}$

11 $\int_0^a \frac{4}{1+x^2}\,dx = \pi$

$4\int_0^a \frac{4}{1+x^2}\,dx = \pi$

$4\left[\tan^{-1}(x)\right]_0^a = \pi$

$\tan^{-1}(a) - \tan^{-1}(0) = \frac{\pi}{4}$

$\tan^{-1}(a) = \frac{\pi}{4}$

$a = \tan\left(\frac{\pi}{4}\right)$

$a = 1$

12 $\int_0^a \frac{4}{4-x^2}\,dx = -\log_e(3)$

$\frac{4}{4-x^2} = \frac{4}{(2-x)(2+x)}$

$= \frac{A}{2-x} + \frac{B}{2+x}$

$= \frac{A(2+x)+B(2-x)}{(2-x)(2+x)}$

So $4 = A(2+x) + B(2-x)$

Let $x = -2$, $\quad 4 = 4B$

$B = 1$

Let $x = 2$, $\quad 4 = 4A$

$A = 1$

So $\int_0^a \left(\frac{1}{2-x} + \frac{1}{2+x}\right)dx = -\log_e(3)$

$\left[-\log_e(2-x) + \log_e(2+x)\right]_0^a = -\log_e(3)$

$\left[\log_e\left(\frac{2+x}{2-x}\right)\right]_0^a = -\log_e(3)$

$\log_e\left(\frac{2+a}{2-a}\right) + \log_e\left(\frac{2}{2}\right) = -\log_e(3)$

$\log_e\left(\frac{2+a}{2-a}\right) + \log_e(1) = -\log_e(3)$

$\log_e\left(\frac{2+a}{2-a}\right) = \log_e\left(\frac{1}{3}\right)$

$\frac{2+a}{2-a} = \frac{1}{3}$

$6 + 3a = 2 - a$

$4a = -4$

$a = -1$

13 $\int_{-1}^a 3\sqrt{x+1}\,dx = 6\sqrt{3}$

$3\int_{-1}^a (x+1)^{\frac{1}{2}}\,dx = 6\sqrt{3}$

$3\left[\frac{2}{3}(x+1)^{\frac{3}{2}}\right]_{-1}^a = 6\sqrt{3}$

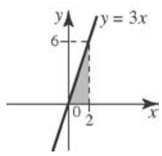
$2\left[(a+1)^{\frac{3}{2}} - (-1+1)^{\frac{3}{2}}\right] = 6\sqrt{3}$

$$(a+1)^{\frac{3}{2}} = 3\sqrt{3}$$
$$(a+1)^{\frac{3}{2}} = (3)^{\frac{3}{2}}$$
$$a + 1 = 3$$
$$a = 2$$

14
$$\int_{-a}^{a} \frac{1}{\sqrt{4-x^2}}\,dx = \frac{\pi}{2}$$
$$\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-a}^{a} = \frac{\pi}{2}$$
$$\sin^{-1}\left(\frac{a}{2}\right) - \sin^{-1}\left(\frac{-a}{2}\right) = \frac{\pi}{2}$$
$$\sin^{-1}\left(\frac{a}{2}\right) + \sin^{-1}\left(\frac{a}{2}\right) = \frac{\pi}{2}$$
$$2\sin^{-1}\left(\frac{a}{2}\right) = \frac{\pi}{2}$$
$$\sin^{-1}\left(\frac{a}{2}\right) = \frac{\pi}{4}$$
$$\frac{a}{2} = \sin\left(\frac{\pi}{4}\right)$$
$$\frac{a}{2} = \frac{1}{\sqrt{2}}$$
$$a = \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$a = \frac{2\sqrt{2}}{2}$$
$$a = \sqrt{2}$$

Exercise 4H — Volumes of solids of revolution

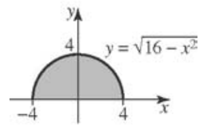
1 a



b $y = 3x$
$$V = \pi\int_0^2 (3x)^2\,dx$$
$$= 9\pi\int_0^2 x^2\,dx$$
$$= 9\pi\left[\frac{1}{3}x^3\right]_0^2$$
$$= 3\pi\left[x^3\right]_0^2$$
$$= 3\pi\left[2^3 - 0^3\right]$$
$= 24\pi$ cubic units

c The solid generated is a cone of maximum radius $r = 6$ and height $h = 2$.
The volume of a cone is given by
$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(6)^2 \times 2$$
$$= \frac{72}{3}\pi$$
$= 24\pi$ cubic units, as required.

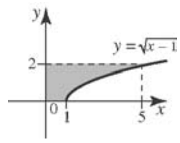
2 a



$$y = \sqrt{16-x^2}$$
$$V = \pi\int_{-4}^{4} (\sqrt{16-x^2})^2\,dx$$
$$= \pi\int_{-4}^{4} (16-x^2)\,dx$$
$$= \pi\left[16x - \frac{1}{3}x^3\right]_{-4}^{4}$$
$$= \pi\left[\left(16\times4 - \frac{4^3}{3}\right) - \left(16\times(-4) - \frac{(-4)^3}{3}\right)\right]$$
$$= \pi\left[2\times16\times4 - \frac{2\times4^3}{3}\right]$$
$$= \pi\left[128 - \frac{128}{3}\right]$$
$$= \frac{\pi}{3}[3\times128 - 128]$$
$= 256\frac{\pi}{3}$ cubic units

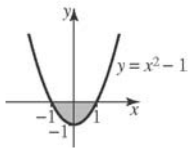
b The solid generated is a sphere of radius $r = 4$. The volume of a sphere is given by
$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi(4)^3$$
$= \frac{256}{3}\pi$ cubic units, as required

3 a



b $y = \sqrt{x-1}$
$$y^2 = x - 1$$
$$x = y^2 + 1$$
$$V = \pi\int_0^2 (y^2+1)^2\,dy$$
$$= \pi\int_0^2 (y^4 + 2y^2 + 1)\,dy$$
$$= \pi\left[\frac{1}{5}y^5 + \frac{2}{3}y^3 + y\right]_0^2$$
$$= \pi\left[\left(\frac{1}{5}(2)^5 + \frac{2}{3}(2)^3 + 2\right) - \left(\frac{1}{5}(0)^5 + \frac{2}{3}(0)^3 + 0\right)\right]$$
$$= \pi\left[\frac{32}{5} + \frac{16}{3} + 2\right]$$
$$= \pi\left[\frac{32\times3 + 16\times5 + 2\times15}{15}\right]$$
$$= \frac{\pi}{15}[96 + 80 + 30]$$
$= \frac{206\pi}{15}$ cubic units

4 a



$y = x^2 - 1$

$V = \pi\int_{-1}^{1}(x^2-1)^2\,dx$

$= \pi\int_{-1}^{1}(x^4-2x^2+1)^2\,dx$

$= \pi\left[\frac{1}{5}x^5-\frac{2}{3}x^3+x\right]_{-1}^{1}$

$= \pi\left[\left(\frac{1}{5}(1)^5-\frac{2}{3}(1)^3+1\right)-\left(\frac{1}{5}(-1)^5-\frac{2}{3}(-1)^3-1\right)\right]$

$= \pi\left[\frac{1}{5}-\frac{2}{3}+1+\frac{1}{5}-\frac{2}{3}+1\right]$

$= 2\pi\left[\frac{1}{5}-\frac{2}{3}+1\right]$

$= 2\pi\left[\frac{3-10+15}{15}\right]$

$= \frac{2\pi}{15}\times 8$

$= \frac{16\pi}{15}$ cubic units

b $y = x^2 - 1$

$y + 1 = x^2$

$x = \pm\sqrt{y+1}$

$V = \pi\int_{-1}^{0}(\pm\sqrt{y+1})^2\,dy$

$= \pi\int_{-1}^{0}(y+1)\,dy$

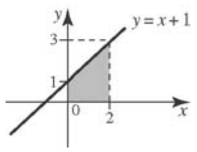
$= \pi\left[\frac{1}{2}y^2+y\right]_{-1}^{0}$

$= \pi\left[\left(\frac{1}{2}(0)^2+0\right)-\left(\frac{1}{2}(-1)^2-1\right)\right]$

$= -\pi\left(\frac{1}{2}-1\right)$

$= \frac{\pi}{2}$ cubic units

5 a i



ii $y = x + 1$

$V = \pi\int_{0}^{2}(x+1)^2\,dx$

$= \pi\int_{0}^{2}(x^2+2x+1)\,dx$

$= \pi\left[\frac{1}{3}x^3+x^2+x\right]_{0}^{2}$

$= \pi\left[\left(\frac{2^3}{3}+2^2+2\right)-\left(\frac{0^3}{3}+0^2+0\right)\right]$

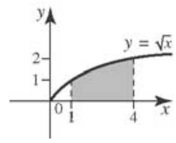
$= \pi\left[\frac{8}{3}+4+2\right]$

$= \pi\left[\frac{8}{3}+6\right]$

$= \pi\left[\frac{8+18}{3}\right]$

$= \frac{26\pi}{3}$ cubic units

b i



ii $y = \sqrt{x}$

$V = \pi\int_{1}^{4}(\sqrt{x})^2\,dx$

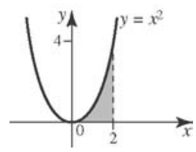
$= \pi\int_{1}^{4}x\,dx$

$= \pi\left[\frac{1}{2}x^2\right]_{1}^{4}$

$= \frac{\pi}{2}\left[4^2-1^2\right]$

$= \frac{15\pi}{2}$ cubic units

c i



ii $y = x^2$

$V = \pi\int_{0}^{2}(x^2)^2\,dx$

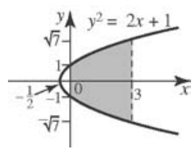
$= \pi\int_{0}^{2}x^4\,dx$

$= \pi\left[\frac{1}{5}x^5\right]_{0}^{2}$

$= \frac{\pi}{5}\left[2^5-0^5\right]$

$= \frac{32\pi}{5}$ cubic units

d i



ii $y^2 = 2x + 1$

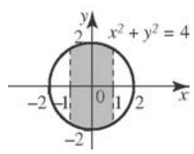
$V = \pi\int_{0}^{3}(2x+1)\,dx$

$= \pi\left[x^2+x\right]_{0}^{3}$

$= \pi\left[(3^2+3)-(0^2+0)\right]$

$= 12\pi$ cubic units

e i



ii $x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$V = \pi\int_{-1}^{1}(4-x^2)\,dx$

$= \pi\left[4x-\frac{1}{3}x^3\right]_{-1}^{1}$

$= \pi\left[\left(4\times1-\frac{(1)^3}{3}\right)-\left(4\times(-1)-\frac{(-1)^3}{3}\right)\right]$

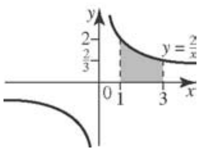
$= \pi\left[4-\frac{1}{3}+4-\frac{1}{3}\right]$

$= \pi\left[8-\frac{2\pi}{3}\right]$

$= (24-2)\frac{\pi}{3}$

$= \frac{22\pi}{3}$ cubic units

f i



ii $y = \frac{2}{x}$

$V = \pi\int_{1}^{3}\left(\frac{2}{x}\right)^2 dx$

$= 4\pi\int_{1}^{3}x^{-2}\,dx$

$= 4\pi\left[-x^{-1}\right]_{1}^{3}$

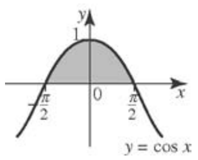
$= -4\pi\left[\frac{1}{x}\right]_{1}^{3}$

$= -4\pi\left[\frac{1}{3}-\frac{1}{1}\right]$

$= 4\pi\left[1-\frac{1}{3}\right]$

$= \frac{8\pi}{3}$ cubic units

g i



ii $y = \cos(x)$

$V = \pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(\cos(x))^2\,dx$

$= \frac{\pi}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(1+\cos(2x))\,dx$

$= \frac{\pi}{2}\left[x+\frac{1}{2}\sin(2x)\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$

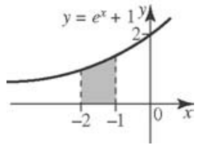
$= \frac{\pi}{2}\left[\left(\frac{\pi}{2}+\frac{1}{2}\sin(\pi)\right)-\left(\frac{-\pi}{2}+\frac{1}{2}\sin(-\pi)\right)\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{2}+0+\frac{\pi}{2}+0\right]$

$= \frac{\pi}{2}\times\pi$

$= \frac{\pi^2}{2}$ cubic units

h i



ii $y = e^x + 1$

$V = \pi\int_{-2}^{-1}(e^x+1)^2\,dx$

$= \pi\int_{-2}^{-1}(e^{2x}+2e^x+1)\,dx$

$= \pi\left[\frac{1}{2}e^{2x}+2e^x+x\right]_{-2}^{-1}$

$= \pi\left[\left(\frac{1}{2}e^{-2}+2e^{-1}-1\right)-\left(\frac{1}{2}e^{-4}+2e^{-2}-2\right)\right]$

$= \pi\left[\frac{1}{2}e^{-2}+2e^{-1}-1-\frac{1}{2}e^{-4}-2e^{-2}+2\right]$

$= \pi\left[\frac{1}{2}e^{-2}+2e^{-1}-\frac{1}{2}e^{-4}-2e^{-2}+1\right]$

≈ 4.787

6 a $y = x + 1 \quad x = 2$

$x = y - 1$

$V = \pi\int_{0}^{1}(2)^2\,dy+\pi\int_{1}^{3}\left[(2)^2-(y-1)^2\right]dy$

$= \pi\int_{0}^{1}4\,dy+\pi\int_{1}^{3}(4-y^2+2y-1)\,dy$

$= 4\pi\int_{0}^{1}1\,dy+\pi\int_{1}^{3}(3+2y-y^2)\,dy$

$= 4\pi[y]_{0}^{1}+\pi\left[3y+y^2-\frac{1}{3}y^3\right]_{1}^{3}$

$= 4\pi[1-0]+\pi\left[\left(3\times3+3^2-\frac{3^3}{3}\right)-\left(3\times1+1^2-\frac{1^3}{3}\right)\right]$

$= 4\pi+\pi\left[9+9-9-3-1+\frac{1}{3}\right]$

$= 4\pi+\pi\left[5+\frac{1}{3}\right]$

$= \frac{12\pi}{3}+\frac{16\pi}{3}$

$= \frac{28\pi}{3}$ cubic units

b $y = \sqrt{x} \quad x = 4 \quad x = 1$

$x = y^2$

$V = \pi\int_{0}^{1}(4^2-1^2)\,dy+\pi\int_{1}^{2}\left[4^2-(y^2)^2\right]dy$

$= \pi\int_{0}^{1}15\,dy+\pi\int_{1}^{2}(16-y^4)\,dy$

$= \pi[15y]_{0}^{1}+\pi\left[16y-\frac{1}{5}y^5\right]_{1}^{2}$

$= 15\pi+\pi\left[\left(16\times2-\frac{2^5}{5}\right)-\left(16-\frac{1^5}{5}\right)\right]$

$= 15\pi+\pi\left[32-\frac{32}{5}-16+\frac{1}{5}\right]$

$= 15\pi + \pi\left[16 - \frac{31}{5}\right]$

$= \left(31 - \frac{31}{5}\right)\pi$

$= \frac{(155-31)}{5}\pi$

$= \frac{124\pi}{5}$ cubic units

c $y = x^2 \quad x = 2$

$x^2 = y$

$V = \pi\int_0^4 (2^2 - y)\,dy$

$= \pi\int_0^4 (4-y)\,dy$

$= \pi\left[4y - \frac{1}{2}y^2\right]_0^4$

$= \pi\left[\left(4\times4 - \frac{4^2}{2}\right) - \left(4\times0 - \frac{0^2}{2}\right)\right]$

$= \pi[16 - 8]$

$= 8\pi$ cubic units

d $y^2 = 2x + 1 \quad x = 3$

$x = \frac{y^2-1}{2}$

$V = \pi\int_{-\sqrt{7}}^{-1}\left[3^2 - \left(\frac{y^2-1}{2}\right)^2\right]dy + \pi\int_{-1}^{1} 3^2\,dy + \pi\int_1^{\sqrt{7}}\left[3^2 - \left(\frac{y^2-1}{2}\right)^2\right]dy$

$= \pi\int_{-\sqrt{7}}^{-1}\left(9 - \frac{y^4-2y^2+1}{4}\right)dy + 9\pi\int_{-1}^{1} 1\,dy + \pi\int_1^{\sqrt{7}}\left(9 - \frac{y^4-2y^2+1}{4}\right)dy$

$= \frac{\pi}{4}\int_{-\sqrt{7}}^{-1}(36 - y^4 + 2y^2 - 1)\,dy + 9\pi\int_{-1}^{1} 1\,dy + \frac{\pi}{4}\int_1^{\sqrt{7}}(36 - y^4 + 2y^2 - 1)\,dy$

$= \frac{\pi}{4}\int_{-\sqrt{7}}^{-1}(35 + 2y^2 - y^4)\,dy + 9\pi\int_{-1}^{1} 1\,dy + \frac{\pi}{4}\int_1^{\sqrt{7}}(35 + 2y^2 - y^4)\,dy$

$= \frac{\pi}{4}\left[35y + \frac{2}{3}y^3 - \frac{1}{5}y^5\right] + 9\pi[y]_{-1}^{1} + \frac{\pi}{4}\left[35y + \frac{2}{3}y^3 - \frac{1}{5}y^5\right]_1^{\sqrt{7}}$

$= \frac{\pi}{4}\left[\left(-35 - \frac{2}{3} + \frac{1}{5}\right) - \left(-35\sqrt{7} + \frac{2}{3}(-\sqrt{7})^3 - \frac{(-\sqrt{7})^5}{5}\right)\right] + 9\pi[1+1] + \frac{\pi}{4}\left[\left(35\sqrt{7} + \frac{2}{3}(\sqrt{7})^3 - \frac{(\sqrt{7})^5}{5}\right) - \left(35 + \frac{2}{3} - \frac{1}{5}\right)\right]$

$= \frac{\pi}{4}\left[-35 - \frac{2}{3} + \frac{1}{5} + 35\sqrt{7} + \frac{2}{3}(\sqrt{7})^3 - \frac{(\sqrt{7})^5}{5}\right] + 18\pi + \frac{\pi}{4}\left[35\sqrt{7} + \frac{2}{3}(\sqrt{7})^3 - \frac{(\sqrt{7})^5}{5} - 35 - \frac{2}{3} + \frac{1}{5}\right]$

$= 2\times\frac{\pi}{4}\left[35\sqrt{7} + \frac{2}{3}(\sqrt{7})^3 - \frac{(\sqrt{7})^5}{5} - 35 - \frac{2}{3} + \frac{1}{5}\right] + 18\pi$

$= \frac{\pi}{2}\left[35\sqrt{7} + \frac{14}{3}\sqrt{7} - \frac{49}{5}\sqrt{7} - 35 - \frac{2}{3} + \frac{1}{5} + 36\right]$

$= \frac{\pi}{2}\left[\left(35 + \frac{14}{3} - \frac{49}{5}\right)\sqrt{7} + 1 - \frac{2}{3} + \frac{1}{5}\right]$

$= \frac{\pi}{2}\left[\frac{448}{15}\sqrt{7} + \frac{8}{15}\right]$

$= \frac{(224\sqrt{7}+4)\pi}{15}$ cubic uints

≈ 124.96 cubic units

e $x^2 + y^2 = 4 \quad x = 1$

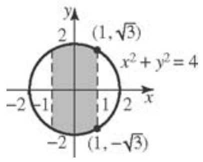
$x^2 = 4 - y^2$

Calculate the point of intersection of line $x = 1$ and circle $x^2 = 4 - y^2$ and mark on diagram

$1^2 = 4 - y^2$

$y^2 = 3$

$y = \pm\sqrt{3}, \quad (1, \pm\sqrt{3})$



$$V = \pi\int_{-2}^{2}(4-y^2)\,\mathrm{d}y - \pi\int_{-\sqrt{3}}^{\sqrt{3}}\left[(4-y^2)-1^2\right]\mathrm{d}y$$

$$= \pi\int_{-2}^{2}(4-y^2)\,\mathrm{d}y - \pi\int_{-\sqrt{3}}^{\sqrt{3}}(3-y^2)\,\mathrm{d}y$$

$$= \pi\int_{-2}^{2}(4-y^2)\,\mathrm{d}y + \pi\int_{-\sqrt{3}}^{\sqrt{3}}(y^2-3)\,\mathrm{d}y$$

$$= \pi\left[4y-\frac{1}{3}y^3\right]_{-2}^{2} + \pi\left[\frac{1}{3}y^3-3y\right]_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \pi\left[\left(4\times2-\frac{2^3}{3}\right)-\left(4\times(-2)-\frac{(-2)^3}{3}\right)\right] + \pi\left[\left(\frac{(\sqrt{3})^3}{3}-3\sqrt{3}\right)-\left(\frac{(-\sqrt{3})^3}{3}+3\sqrt{3}\right)\right]$$

$$= \pi\left[8-\frac{8}{3}+8-\frac{8}{3}\right]+\pi\left[\frac{3\sqrt{3}}{3}-3\sqrt{3}+\frac{3\sqrt{3}}{3}-3\sqrt{3}\right]$$

$$= \pi\left[16-\frac{16}{3}\right]+\pi\left[\sqrt{3}-3\sqrt{3}+\sqrt{3}-3\sqrt{3}\right]$$

$$= \left(\frac{48-16}{3}\right)\pi - 4\sqrt{3}\pi$$

$$= \frac{(32-12\sqrt{3})\pi}{3} \text{ cubic units}$$

≈ 11.745 cubic units

f $y = \dfrac{2}{x} \quad x = 3 \quad x = 1$

$x = \dfrac{2}{y}$

$$V = \pi\int_{0}^{\frac{2}{3}}(3^2-1^2)\,\mathrm{d}y + \pi\int_{\frac{2}{3}}^{2}\left[\left(\frac{2}{y}\right)^2-1^2\right]\mathrm{d}y$$

$$= \pi\int_{0}^{\frac{2}{3}}8\,\mathrm{d}y + \pi\int_{\frac{2}{3}}^{2}\left(\frac{4}{y^2}-1\right)\mathrm{d}y$$

$$= 8\pi\int_{0}^{\frac{2}{3}}1\,\mathrm{d}y + \pi\int_{\frac{2}{3}}^{2}(4y^{-2}-1)\,\mathrm{d}y$$

$$= 8\pi[y]_{0}^{\frac{2}{3}} + \pi\left[-4y^{-1}-y\right]_{\frac{2}{3}}^{2}$$

$$= 8\pi\left[\frac{2}{3}-0\right] - \pi\left[\frac{4}{y}+y\right]_{\frac{2}{3}}^{2}$$

$$= \frac{16\pi}{3} - \pi\left[\left(\frac{4}{2}+2\right)-\left(\frac{4}{\frac{2}{3}}+\frac{2}{3}\right)\right]$$

$$= \frac{16\pi}{3} - \pi\left[4-\frac{12}{2}-\frac{2}{3}\right]$$

$$= \frac{16\pi}{3} - \pi\left(-2\pi-\frac{2}{3}\right)$$

$$= \frac{16\pi}{3} + \frac{2\pi}{3} + 2\pi$$

$$= \frac{18\pi}{3} + 2\pi$$

$= 6\pi + 2\pi$

$= 8\pi$ cubic units.

7 a $y = x^2 + 2 \quad y = 4 - x^2$

$x^2 + 2 = 4 - x^2$

$2x^2 = 2$

$x^2 = 1$

$x = \pm 1$

$y = (\pm 1)^2 + 2$

$= 1 + 2$

$= 3$

The points of intersection are (1, 3) and (−1, 3)

$\therefore$ **B**

b $V = \pi\int_{-1}^{1}\left[(4-x^2)^2-(x^2+2)^2\right]dx$

$= \pi\int_{-1}^{1}\left[16-8x^2+x^4-(x^4+4x^2+4)\right]dx$

$= \pi\int_{-1}^{1}(16-8x^2+x^4-x^4-4x^2-4)\,dx$

$= \pi\int_{-1}^{1}(12-12x^2)\,dx$

$\therefore$ **D**

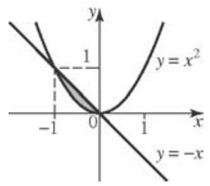
c $y = x^2 + 2 \qquad y = 4 - x^2$

$x^2 = y - 2 \qquad x^2 = 4 - y$

$V = \pi\int_{3}^{4}(4-y)\,dy + \pi\int_{2}^{3}(y-2)\,dy$

$\therefore$ **A**

8 a



$y = x^2 \quad y = -x$

$V = \pi\int_{-1}^{0}\left[(-x)^2-(x^2)^2\right]dx$

$= \pi\int_{-1}^{0}(x^2-x^4)\,dx$

$= \pi\left[\frac{1}{3}x^3-\frac{1}{5}x^5\right]_{-1}^{0}$

$= \pi\left[\left(\frac{1}{3}(0)^3-\frac{1}{5}(0)^5\right)-\left(\frac{1}{3}(-1)^3-\frac{1}{5}(-1)^5\right)\right]$

$= \pi\left[\frac{1}{3}-\frac{1}{5}\right]$

$= \pi\frac{(5-3)}{15}$

$= \frac{2\pi}{15}$ cubic units

b $y = x^2 \quad y = -x$

$x^2 = y \quad x = -y$

$V = \pi\int_{0}^{1}\left[y-(-y)^2\right]dy$

$= \pi\int_{0}^{1}(y-y^2)\,dy$

$= \pi\left[\frac{1}{2}y^2-\frac{1}{3}y^3\right]_{0}^{1}$

$= \pi\left[\left(\frac{1}{2}-\frac{1}{3}\right)-0\right]$

$= \frac{\pi(3-2)}{6}$

$= \frac{\pi}{6}$ cubic units

9 $y = \sec(x) \quad x = \frac{\pi}{4}$

$y = \sec\left(\frac{\pi}{4}\right)$

$= \frac{1}{\cos\left(\frac{\pi}{4}\right)}$

$= \frac{1}{\frac{1}{\sqrt{2}}}$

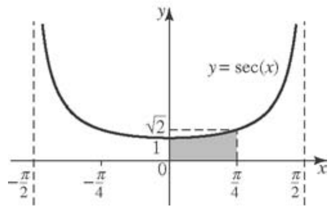
$= \sqrt{2}, \left(\frac{\pi}{4}, \sqrt{2}\right)$

$x = 0, \quad y = \sec(0)$

$= \frac{1}{\cos(0)}$

$= \frac{1}{1}$

$= 1, \quad (0, 1)$



$V = \pi\int_{0}^{\frac{\pi}{4}}(\sec(x))^2\,dx$

$= \pi\left[\tan(x)\right]_{0}^{\frac{\pi}{4}}$

$= \pi\left[\tan\left(\frac{\pi}{4}\right)-\tan(0)\right]$

$= \pi[1-0]$

$= \pi$ cubic units

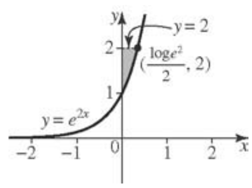
10 $y = e^{2x} \quad y = 2$

$e^{2x} = 2$

$2x = \log_e(2)$

$x = \frac{\log_e(2)}{2} \approx 0.3467$

Point of intersection $\left(\frac{\log_e(2)}{2}, 2\right)$



$y = e^{2x}$

$\log_e(y) = 2x$

$x = \frac{\log_e(y)}{2}$

$V = \pi\int_{1}^{2}\left(\frac{\log_e(y)}{2}\right)^2 dy$

$= \frac{\pi}{4}\int_{1}^{2}(\log_e(y))^2\,dy$

$= \int_{a}^{b}u\frac{dv}{dx}dy = [uv]_{a}^{b} - \int_{a}^{b}v\frac{du}{dy}dy$

Let $\quad u = (\log_e(y))^2 \quad$ and $\quad \frac{dv}{dy} = 1$

So $\dfrac{du}{dy} = \dfrac{2\log_e(y)}{y}$ and $v = y$

So $V = \pi\left[(\log_e(y))^2 y\right]_1^2 - \pi\int_1^2 y\left(\dfrac{2\log_e(y)}{y}\right)dy$

$= \pi\left[2(\log_e(2))^2 - (\log_e(1))^2\right] - 2\pi\int_1^2 \log_e(y)\,dy$

$= 2\pi(\log_e(2))^2 - 2\pi\int_1^2 \log_e(y)\,dy$

Let $u = \log_e(y)$ and $\dfrac{dv}{dy} = 1$

So $\dfrac{du}{dy} = \dfrac{1}{y}$ and $v = y$

So $V = 2\pi(\log_e(2))^2 - 2\pi\left\{\left[y\log_e(y)\right]_1^2 - \int_1^2 \dfrac{y}{y}dy\right\}$

$= 2\pi(\log_e(2))^2 - 2\pi\left[2\log_e(2) - \log_e(1)\right] + 2\pi\int_1^2 1\,dy$

$= 2\pi(\log_e(2))^2 - 4\pi\log_e(2) + 2\pi\left[y\right]_1^2$

$= 2\pi(\log_e(2))^2 - 4\pi\log_e(2) + 2\pi\left[2-1\right]$

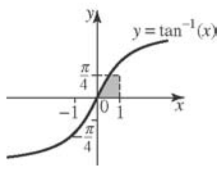
$= 2\pi(\log_e(2))^2 - 4\pi\log_e(2) + 2\pi$

$= 2\pi(\log_e(2) - 1)^2 \approx 0.592$ cubic units.

11 $y = \tan^{-1}(x) \quad x = 1$

$y = \tan^{-1}(1)$

$y = \dfrac{\pi}{4}, \quad \left(1, \dfrac{\pi}{4}\right)$



$y = \tan^{-1}(x)$

$x = \tan(y)$

$V = \pi\int_0^{\frac{\pi}{4}}\left[1^2 - (\tan(y))^2\right]dy$

$= \pi\int_0^{\frac{\pi}{4}}(1 - \tan^2(y))\,dy$

$= \pi\int_0^{\frac{\pi}{4}}\left[1 - (\sec^2(y) - 1)\right]dy$

$= \pi\int_0^{\frac{\pi}{4}}(2 - \sec^2(y))\,dy$

$= \pi\left[2y - \tan(y)\right]_0^{\frac{\pi}{4}}$

$= \pi\left[\left(\dfrac{2\pi}{4} - \tan\left(\dfrac{\pi}{4}\right)\right) - (2\times 0 - \tan(0))\right]$

$= \pi\left(\dfrac{\pi}{2} - 1\right)$

$= \dfrac{\pi^2}{2} - \pi$ cubic units

≈ 1.793 cubic units

12 $y = 2 - \dfrac{x^2}{6} \quad x = 1 \quad x = -1$

$x = 1, \quad y = 2 - \dfrac{1^2}{6}$

$y = \dfrac{11}{6}, \left(1, \dfrac{11}{6}\right)$

$x = -1, \quad y = 2 - \dfrac{(-1)^2}{6}$

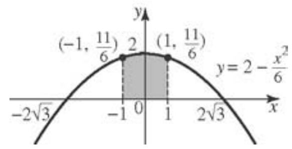
$= \dfrac{11}{6}, \left(-1, \dfrac{11}{6}\right)$

$x = 0, \quad y = 2, \quad (0, 2)$

$y = 0, \quad 0 = 2 - \dfrac{x^2}{6}$

$x^2 = 12$

$x = \pm 2\sqrt{3}, \quad (\pm 2\sqrt{3}, 0)$



$V = \pi\int_{-1}^{1}\left(2 - \dfrac{x^2}{6}\right)^2 dx$

$= \pi\int_{-1}^{1}\left(4 - \dfrac{2x^2}{3} + \dfrac{x^4}{36}\right)dx$

$= \pi\left[4x - \dfrac{2x^3}{9} + \dfrac{x^5}{180}\right]_{-1}^{1}$

$= \pi\left[\left(4 - \dfrac{2}{9} + \dfrac{1}{180}\right) - \left(-4 + \dfrac{2}{9} - \dfrac{1}{180}\right)\right]$

$= \pi\left[4 - \dfrac{2}{9} + \dfrac{1}{180} + 4 - \dfrac{2}{9} + \dfrac{1}{180}\right]$

$= \pi\left[8 - \dfrac{4}{9} + \dfrac{2}{180}\right]$

$= \dfrac{\pi}{90}(8\times 90 - 4\times 10 + 1)$

$= \dfrac{\pi}{90}(720 - 40 + 1)$

$= \dfrac{\pi}{90}(681)$

$= \dfrac{227\pi}{30}$ cubic units

13 **a** $x = 1, \quad y = \dfrac{1}{\sqrt{4 - x^2}}$

$= \dfrac{1}{\sqrt{4 - 1^2}}$

$= \dfrac{1}{\sqrt{3}}$

A has coordinates $\left(1, \dfrac{1}{\sqrt{3}}\right)$

b $V = \pi\int_0^1\left(\dfrac{1}{\sqrt{4 - x^2}}\right)^2 dx$

$= \pi\int_0^1 \dfrac{1}{4 - x^2}dx$

$\dfrac{1}{4 - x^2} = \dfrac{1}{(2-x)(2+x)}$

$= \dfrac{a}{2-x} + \dfrac{b}{2+x}$

$= \dfrac{a(2+x) + b(2-x)}{(2-x)(2+x)}$

So $1 = a(2+x) + b(2-x)$

Let $x = -2, \quad 1 = 4b$

$b = \frac{1}{4}$

Let $x = 2, \quad 1 = 4a$

$a = \frac{1}{4}$

So $V = \pi\int_0^1\left(\frac{1}{4(2-x)}+\frac{1}{4(2+x)}\right)dx$

$= \frac{\pi}{4}\int_0^1\left(\frac{1}{(2-x)}+\frac{1}{(2+x)}\right)dx$

$= \frac{\pi}{4}[-\log_e(2-x)+\log_e(2+x)]_0^1$

$= \frac{\pi}{4}\left[\log_e\left(\frac{2+x}{2-x}\right)\right]_0^1$

$= \frac{\pi}{4}\left[\log_e\left(\frac{3}{1}\right)-\log_e\left(\frac{2}{2}\right)\right]$

$= \frac{\pi}{4}[\log_e(3)-\log_e(1)]$

$= \frac{\pi}{4}\log_e(3)$ cubic units.

c $y = \frac{1}{\sqrt{4-x^2}} \quad x = 1$

$y^2 = \frac{1}{\sqrt{4-x^2}}$

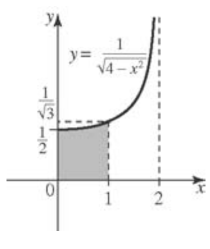
$4 - x^2 = \frac{1}{y^2}$

$x^2 = 4-\frac{1}{y^2}$

$x = 0, \quad y = \frac{1}{\sqrt{4-0^2}}$

$= \frac{1}{\sqrt{4}}$

$= \frac{1}{2}, \quad \left(0, \frac{1}{2}\right)$



$y = \frac{1}{\sqrt{4-x^2}}$

$y^2 = \frac{1}{\sqrt{4-x^2}}$

$4 - x^2 = \frac{1}{y^2}$

$x^2 = 4-\frac{1}{y^2}$

$V = \pi\int_0^{\frac{1}{2}}1^2\,dy+\pi\int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}}\left[1^2-\left(4-\frac{1}{y^2}\right)\right]dy$

$= \pi\int_0^{\frac{1}{2}}1\,dy+\pi\int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}}\left(-3+\frac{1}{y^2}\right)dy$

$= \pi\int_0^{\frac{1}{2}}1\,dy+\pi\int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}}(y^{-2}-3)\,dy$

$= \pi[y]_0^{\frac{1}{2}}+\pi\left[-y^{-1}-3y\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}}$

$= \pi\left[\frac{1}{2}-0\right]-\pi\left[y^{-1}+3y\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}}$

$= \frac{\pi}{2}-\pi\left\{\left[\left(\frac{1}{\sqrt{3}}\right)^{-1}+\frac{3}{\sqrt{3}}\right]-\left[\left(\frac{1}{2}\right)^{-1}+\frac{3}{2}\right]\right\}$

$= \frac{\pi}{2}-\pi\left(\sqrt{3}+\frac{3\sqrt{3}}{3}-2-\frac{3}{2}\right)$

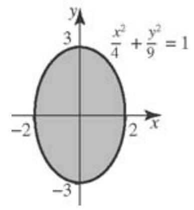
$= \frac{\pi}{2}-\pi\left(2\sqrt{3}-\frac{7}{2}\right)$

$= \left(\frac{1}{2}-2\sqrt{3}+\frac{7}{2}\right)\pi$

$= (4-2\sqrt{3})\pi$ cubic units

14 a $\frac{x^2}{4}+\frac{y^2}{9} = 1$

Ellipse: (2, 0), (–2, 0), (0, 3), (0, –3)
centre (0, 0)



$\frac{x^2}{4}+\frac{y^2}{9} = 1$

$\frac{y^2}{9} = 1-\frac{x^2}{4}$

$y^2 = 9-\frac{9x^2}{4}$

$V = \pi\int_{-2}^{2}\left(9-\frac{9x^2}{4}\right)dx$

$= \pi\left[9x-\frac{3x^3}{4}\right]_{-2}^{2}$

$= \pi\left[\left(9\times 2-\frac{3(2)^3}{4}\right)-\left(9\times(-2)-\frac{3(-2)^3}{4}\right)\right]$

$= \pi\left[18-\frac{24}{4}+18-\frac{24}{4}\right]$

$= \pi[36 - 6 - 6]$

$= 24\pi$ cubic units

b $\frac{x^2}{4}+\frac{y^2}{9} = 1$

$\frac{x^2}{4} = 1-\frac{y^2}{9}$

$x^2 = 4-\frac{4y^2}{9}$

$V = \pi\int_{-3}^{3}\left(4-\frac{4y^2}{9}\right)dy$

$= \pi\left[4y-\frac{4y^3}{27}\right]_{-3}^{3}$

$= \pi\left[\left(4\times3-\frac{4(3)^3}{27}\right)-\left(4\times(-3)-\frac{4(-3)^3}{27}\right)\right]$

$= \pi[12 - 4 + 12 - 4]$

$= \pi[24 - 8]$

$= 16\pi$ cubic units

15 a $y = x^2$ and $y = \sqrt{8x}$

$x^2 = \sqrt{8x}$

$x^4 = 8x$

$x^4 - 8x = 0$

$x(x^3 - 8) = 0$

$x = 0$ $\quad x^3 - 8 = 0$

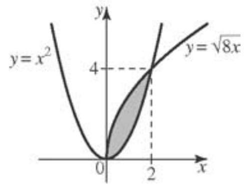
$y = 0^2$ $\quad x^3 = 8$

$= 0$ $\quad x = 2$

$y = (2)^2$

$= 4$

Intersection at (0, 0) and (2, 4)



$V = \pi\int_0^2\left[(\sqrt{8x})^2 - (x^2)^2\right]dx$

$= \pi\int_0^2(8x - x^4)\,dx$

$= \pi\left[4x^2 - \frac{1}{5}x^5\right]_0^2$

$= \pi\left[\left(4(2)^2 - \frac{2^5}{5}\right)-\left(4(0)^2 - \frac{0^5}{5}\right)\right]$

$= \pi\left[16 - \frac{32}{5}\right]$

$= \frac{\pi}{5}[16\times5 - 32]$

$= \frac{\pi}{5}[80 - 32]$

$= \frac{48\pi}{5}$ cubic units

b $y = x^2$ $\quad y = \sqrt{8x}$

$x^2 = y$ $\quad y^2 = 8x$

$x = \frac{y^2}{8}$

$V = \pi\int_0^4\left[y - \left(\frac{y^2}{8}\right)^2\right]dy$

$= \pi\int_0^4\left(y - \frac{y^4}{64}\right)dy$

$= \pi\left[\frac{1}{2}y^2 - \frac{1}{320}y^5\right]_0^4$

$= \pi\left[\left(\frac{4^2}{2} - \frac{4^5}{320}\right)-\left(\frac{0^2}{2} - \frac{0^5}{320}\right)\right]$

$= \pi[8 - 3.2]$

$= 4.8\pi$ cubic units

16 a $y = x^3$ and $y = x^2$

$x^3 = x^2$

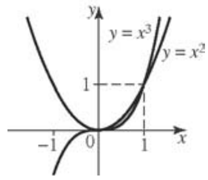
$x^3 - x^2 = 0$

$x^2(x - 1) = 0$

$x = 0$ $\quad x = 1$

$y = 0$ $\quad y = 1$

Intersection at (0, 0) and (1, 1)



$V = \pi\int_0^1\left[(x^2)^2 - (x^3)^2\right]dx$

$= \pi\int_0^1(x^4 - x^6)\,dx$

$= \pi\left[\frac{1}{5}x^5 - \frac{1}{7}x^7\right]_0^1$

$= \pi\left[\left(\frac{1}{5} - \frac{1}{7}\right) - 0\right]$

$= \frac{\pi}{35}[7 - 5]$

$= \frac{2\pi}{35}$ cubic units

b $y = x^3$ $\quad y = x^2$

$x = y^{\frac{1}{3}}$ $\quad x = y^{\frac{1}{2}}$

$V = \pi\int_0^1\left[\left(y^{\frac{1}{3}}\right)^2 - \left(y^{\frac{1}{2}}\right)^2\right]dy$

$= \pi\int_0^1\left(y^{\frac{2}{3}} - y\right)dy$

$= \pi\left[\left(\frac{3}{5}y^{\frac{5}{3}} - \frac{1}{2}y^2\right)\right]_0^1$

$= \pi\left[\left(\frac{3}{5} - \frac{1}{2}\right) - 0\right]$

$= \frac{\pi}{10}[6 - 5]$

$= \frac{\pi}{10}$ cubic units.

Exercise 4I — Approximation using Simpson's rule

1 a $\int_0^{\frac{\pi}{2}}\sin(x)^2\,dx$, 4 strips

$w = \frac{b-a}{n}$ where $a = 0$, $b = \frac{\pi}{2}$ and $n = 4$

So $w = \frac{\frac{\pi}{2} - 0}{4}$

$= \frac{\pi}{8}$

n	x_n	y_n
0	0	0
1	$\frac{\pi}{8}$	0.154
2	$\frac{2\pi}{8}$	0.578
3	$\frac{3\pi}{8}$	0.983
4	$\frac{4\pi}{8}$	0.624

By Simpson's rule,

Area $\approx \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + ...) + 2(y_2 + y_4 + ...)]$

So $\int_0^{\frac{\pi}{2}} \sin(x)^2\, dx \approx \frac{w}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)]$

$= \frac{\pi}{24}[0 + 0.624 + 4(0.154 + 0.983) + 2(0.578)]$

$= 0.828$

b $\int_0^{\frac{\pi}{2}} \sin(x)^2\, dx$, 6 strips

$w = \frac{b-a}{n}$ where $a = 0$, $b = \frac{\pi}{2}$ and $n = 6$

So $w = \frac{\frac{\pi}{2} - 0}{6}$

$= \frac{\pi}{12}$

n	x_n	y_n
0	0	0
1	$\frac{\pi}{12}$	0.068
2	$\frac{2\pi}{12}$	0.271
3	$\frac{3\pi}{12}$	0.578
4	$\frac{4\pi}{12}$	0.890
5	$\frac{5\pi}{12}$	0.990
6	$\frac{6\pi}{12}$	0.624

By Simpson's rule,

Area $\approx \frac{w}{3}[y_0 + y_n + 4[y_1 + y_3 + ...] + 2(y_2 + y_4 + ...)]$

So $\int_0^{\frac{\pi}{2}} \sin(x)^2\, dx \approx \frac{w}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

$= \frac{\pi}{36}[0 + 0.624 + 4(0.068 + 0.578 + 0.990) + 2(0.271 + 0.890)]$

$= 0.828$

2 a $\int_0^1 2^{-x^2}\, dx$, 4 strips

$w = \frac{b-a}{n}$ where $a = 0$, $b = 1$ and $n = 4$

So $w = \frac{1-0}{4}$

$= \frac{1}{4}$

n	x_n	y_n
0	0	1
1	$\frac{1}{4}$	0.958
2	$\frac{2}{4}$	0.841
3	$\frac{3}{4}$	0.677
4	$\frac{4}{4}$	0.5

By Simpson's rule,

Area $\approx \frac{w}{3}\left[y_0 + y_n + 4(y_1 + y_3 + ...) + 2(y_2 + y_4 + ...)\right]$

So $\int_0^1 2^{-x^2}\,dx \approx \frac{w}{3}\left[y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)\right]$

$= \frac{1}{12}\left[1 + 0.5 + 4(0.958 + 0.677) + 2(0.841)\right]$

$= 0.810$

b $\int_0^1 2^{-x^2}\,dx$, 6 strips

$w = \frac{b-a}{n}$ where $a = 0$, $b = 1$ and $n = 6$

So $w = \frac{1-0}{6}$

$= \frac{1}{6}$

n	x_n	y_n
0	0	1
1	$\frac{1}{6}$	0.981
2	$\frac{2}{6}$	0.926
3	$\frac{3}{6}$	0.841
4	$\frac{4}{6}$	0.735
5	$\frac{5}{6}$	0.618
6	$\frac{6}{6}$	0.5

By Simpson's rule,

Area $\approx \frac{w}{3}\left[y_0 + y_n + 4(y_1 + y_3 + ...) + 2(y_2 + y_4 + ...)\right]$

So $\int_0^1 2^{-x^2}\,dx \approx \frac{1}{18}\left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$

$= \frac{1}{18}\left[1 + 0.5 + 4(0.981 + 0.841 + 0.618) + 2(0.926 + 0.735)\right]$

$= 0.810$

3 a $\int_0^2 \frac{1}{1+x^2}\,dx$, 4 strips

$w = \frac{b-a}{n}$ where $a = 0$, $b = 2$ and $n = 4$

So $w = \frac{2-0}{4}$

$= \frac{1}{2}$

n	x_n	y_n
0	0	1
1	$\frac{1}{2}$	0.8
2	$\frac{2}{2}$	0.5
3	$\frac{3}{2}$	0.308
4	$\frac{4}{2}$	0.2

By Simpson's rule,

Area $\approx \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + ...) + 2(y_2 + y_4 + ...)]$

So $\int_0^2 \frac{1}{1+x^2}dx \approx \frac{w}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)]$

$= \frac{1}{6}[1 + 0.2 + 4(0.8 + 0.308) + 2(0.5)]$

$= 1.105$

b $\int_0^2 \frac{1}{1+x^2}dx$, 6 strips

$w = \frac{b-a}{n}$ where $a = 0, b = 2$ and $n = 6$

So $w = \frac{2-0}{6}$

$= \frac{1}{3}$

n	x_n	y_n
0	0	1
1	$\frac{1}{3}$	0.9
2	$\frac{2}{3}$	0.692
3	$\frac{3}{3}$	0.5
4	$\frac{4}{3}$	0.36
5	$\frac{5}{3}$	0.265
6	$\frac{6}{3}$	0.2

By Simpson's rule,

Area $\approx \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + ...) + 2(y_2 + y_4 + ...)]$

So $\int_0^2 \frac{1}{1+x^2} \approx \frac{w}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

$= \frac{1}{9}[1 + 0.2 + 4(0.9 + 0.5 + 0.265) + 2(0.692 + 0.36)]$

$= 1.107$

4 a $\int_0^2 \frac{1}{1+x^2}dx$

$= \left[\tan^{-1}(x)\right]_0^2$

$= \tan^{-1}(2) - \tan^{-1}(0)$

$= \tan^{-1}(2)$

≈ 1.107

b This answer is very close to the one obtained in question **3**.

5 a i $\int_{-3}^{3} \sqrt{9-x^2}\,dx$

Let $x = 3\sin(\theta)$

$\frac{dx}{d\theta} = 3\cos(\theta)$ or $dx = 3\cos(\theta)\,d\theta$

When $x = -3$, $-3 = 3\sin(\theta)$

$\sin(\theta) = -1$

$\theta = \frac{-\pi}{2}$

When $x = 3$, $3 = 3\sin(\theta)$

$\sin(\theta) = 1$

$\theta = \frac{\pi}{2}$

So $\int_{-3}^{3}\sqrt{9-x^2}\,\mathrm{d}x$

$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sqrt{9-(3\sin(\theta))^2}\times 3\cos(\theta)\,\mathrm{d}\theta$

$= 3\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sqrt{9-(1\sin^2(\theta))}\cos(\theta)\,\mathrm{d}\theta$

$= 9\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\cos(\theta)\cos(\theta)\,\mathrm{d}\theta$

$= 9\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\cos^2(\theta)\,\mathrm{d}\theta$

$= \frac{9}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(1-\cos(2\theta))\,\mathrm{d}\theta$

$= \frac{9}{2}\left[\theta-\frac{1}{2}\sin(2\theta)\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{9}{2}\left[\left(\frac{\pi}{2}-\frac{1}{2}\sin(\pi)\right)-\left(\frac{-\pi}{2}\right)-\frac{1}{2}\sin(-\pi)\right]$

$= \frac{9}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]$

$= \frac{9\pi}{2}$

≈ 14.137

ii Answers to this question will vary as the solution is an approximation.

$\int_{-3}^{3}\sqrt{9-x^2}\,\mathrm{d}x$, 6 strips

Since a hemisphere is symmetric about the y-axis

$\int_{-3}^{3}\sqrt{9-x^2}\,\mathrm{d}x = 2\int_{0}^{3}\sqrt{9-x^2}\,\mathrm{d}x$

$w = \frac{b-a}{n}$ where $a = 0$, $b = 3$ and $n = 6$

So $w = \frac{3-0}{6}$

$= \frac{1}{2}$

n	x_n	y_n
0	0	3
1	$\frac{1}{2}$	2.958
2	$\frac{2}{2}$	2.828
3	$\frac{3}{2}$	2.598
4	$\frac{4}{2}$	2.236
5	$\frac{5}{2}$	1.658
6	$\frac{6}{2}$	0

By Simpson's rule,

Area $\approx \frac{w}{3}[y_0+y_n+4(y_1+y_3+...)+2(y_2+y_4+...)]$

So $\int_{0}^{3}\sqrt{9-x^2}\,\mathrm{d}x \approx \frac{w}{3}[y_0+y_6+4(y_1+y_3+y_5)+2(y_2+y_4)]$

So $\int_{-3}^{3}\sqrt{9-x^2}\,dx \approx \frac{2w}{3}[y_0+y_6+4(y_1+y_3+y_5)+2(y_2+y_4)]$

$= \frac{1}{3}[3+0+4(2.958+2.598+1.658)+2(2.828+2.236)]$

$= 13.995$

b Percentage difference $= \frac{14.137-13.995}{14.137}\times 100\%$

$= 1\%$

6 width = 60 m
depth = 35 m
Volume = area of cross-section × width
For the area of the cross-section:

Area $\approx \frac{w}{3}[y_0+y_n+4(y_1+y_3+...)+2(y_2+y_4+...)]$

$w = \frac{b-a}{n}$ where $n = 6$, $a = 0$ and $b = 60$ m

So $w = \frac{60-0}{6}$

$= 10$ m

Area of cross-section

$\approx \frac{w}{3}[y_0+y_6+4(y_1+y_3+y_5)+2(y_2+y_4)]$

$= \frac{10}{3}[1+2+4(1.5+4+3)+2(3+5.5)]$

$= 180$ m^2

Volume = 180 × 35

= 6300 m^3

The volume of the rock to be removed is 6300 m^3

7 $w = \frac{b-a}{n}$ where $a = 0$, $b = 30$ m and $n = 6$

So $w = \frac{30-0}{6}$

$= 5$ m

By Simpson's rule,

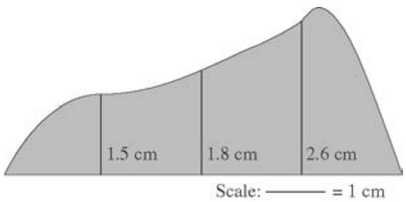
Area $\approx \frac{w}{3}[y_0+y_n+4(y_1+y_3+...)+2(y_2+y_4+...)]$

So Area $\approx \frac{w}{3}[y_0+y_6+4(y_1+y_3+y_5)+2(y_2+y_4)]$

$= \frac{5}{3}[28+18+4(33+27+21)+2(30+23)]$

$= 793.3$ m^2

8 Answers to this question will vary as the solution is an approximation.



n	y_n
0	0
1	1.5
2	1.8
3	2.6
4	0

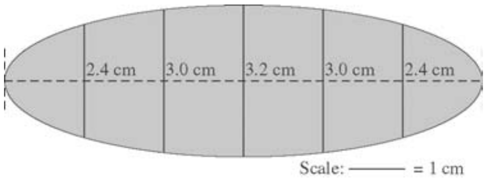
$w = 1.7$ cm

Area $\approx \frac{w}{3}[y_0+y_4+4(y_1+y_3)+2(y_2)]$

$= \frac{1.7}{3}[0+0+4(1.5+2.6)+2(1.8)]$

$= 11.3$ cm^3

9 a Answers to this question will vary as the solution is an approximation.



n	y_n
0	0
1	2.4
2	3.0
3	3.2
4	3.0
5	2.4
6	0

$w = 1.4$ cm

$$\text{Area} \approx \frac{w}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$
$$= \frac{1.4}{3}[0 + 0 + 4(2.4 + 3.2 + 2.4) + 2(3.0 + 3.0)]$$
$$= 20.53 \text{ cm}^2$$

b Area = πab

length of ellipse $= 6w$
$= 6 \times 1.4$ cm
$= 8.4$ cm

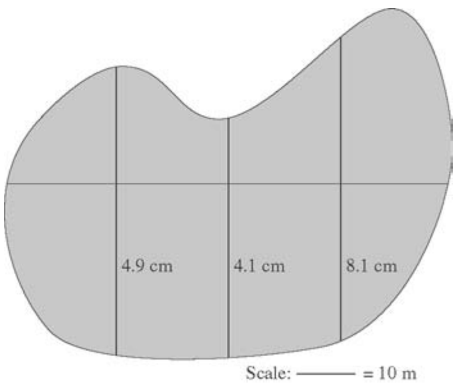
$$a = \frac{8.4}{2} = 4.2 \text{ cm}$$
$$b = \frac{3.2}{2} = 1.6 \text{ cm}$$

Area = πab
$= \pi \times 4.2 \times 1.6$
$= 21.1 \text{ cm}^2$

$$\text{Percentage difference} = \frac{|21.11 - 20.53|}{21.11} \times 100\%$$
$$= 2.7\%$$

The two answers are only 2.7% different, so they are quite close.

10 Answers to this question will vary as the solution is an approximation.



n	y_n
0	0
1	4.9
2	4.1
3	5.1
4	0

$w = 1.9$ m

$$\text{Area} \approx \frac{w}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)]$$
$$= \frac{1.9}{3}[0 + 0 + 4(4.9 + 5.1) + 2(4.1)]$$
$$= 30.53 \text{ cm}^2$$

The scale is 1 cm : 10 m so

$$\text{Area} \approx 30.53 \text{ cm}^2 \times \left(\frac{10 \text{ m}}{1 \text{ cm}}\right)^2$$
$$= 30.53 \text{ cm}^2 \times \frac{100 \text{ m}^2}{1 \text{ cm}^2}$$
$$= 3053 \text{ m}^2$$

11 Show that if $\int_a^b x^2\,dx$ is approximated by Simpson's rule by 2 strips then it gives the exact value.

The exact value is given by integration:

$$\int_a^b x^2\,dx = \left[\frac{1}{3}x^3\right]_a^b$$
$$= \frac{1}{3}\left[x^3\right]_a^b$$
$$= \frac{1}{3}(b^3 - a^3)$$

$$w = \frac{b-a}{n} \text{ where } n = 2$$

So $w = \dfrac{b-a}{2}$

n	x_n	y_n
0	0	0
1	$\frac{b-a}{2}$	$\left(\frac{b-a}{2}\right)^2$
2	$b-a$	$(b-a)^2$

By Simpson's rule,

$$\text{Area} \approx \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$$

So
$$\int_a^b x^2\,dx \approx \frac{(b-a)}{6}\left[0 + (b-a)^2 + 4\left(\frac{b-a}{2}\right)^2\right]$$
$$= \frac{(b-a)}{6}\left[(b-a)^2 + \frac{4}{4}(b-a)^2\right]$$
$$= \frac{(b-a)^3}{6}(1+1)$$
$$= \frac{(b-a)^3}{6} \times 2$$
$$= \frac{(b-a)^3}{3}$$

But this is the exact result obtained previously, so

$\int_a^b x^2\,dx = \dfrac{(b-a)^3}{3}$ meaning Simpson's rule gives the exact value in this case.

12 If $w = x_2 - x_1 = x_1 - x_0$, show that

$$\int_{x_0}^{x_2} (ax^2 + bx + c)\,dx = \frac{w}{3}(y_2 + y_0 + 4y_1)$$

$$\text{LHS} = \int_{x_0}^{x_2} (ax^2 + bx + c)\,dx$$
$$= \left[\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx)\right]_{x_0}^{x_2}$$
$$= \left(\frac{a}{3}x_2^3 + \frac{b}{2}x_2^2 + cx_2\right) - \left(\frac{a}{3}x_0^3 + \frac{b}{2}x_0^2 + cx_0\right)$$

$= \frac{1}{6}(2ax_2^3 + 3bx_2^2 + 6cx_2 - 2ax_0^3 - 3bx_0^2 - 6cx_0)$

$= \frac{1}{6}\left[2a(x_2^3 - x_0^3) + 3b(x_2^2 - x_0^2) + 6c(x_2 - x_0)\right]$

$= \frac{1}{6}\left[2a(x_2 - x_0)(x_2^2 + x_0x_2 + x_0^2) + 3b(x_2 + x_0)(x_2 + x_0) + 6c(x_2 + x_0)\right]$

$= \frac{1}{6}(x_2 - x_0)\left[2a(x_2^2 + x_0x_2 + x_0^2) + 3b(x_2 + x_0) + 6c\right]$

Now $w = \frac{x_2 - x_0}{2}$

$x_2 - x_0 = 2w$

So LHS $= \frac{1}{6}(2w)\left[2a(x_2^2 + x_0x_2 + x_0^2) + 3b(x_2 + x_0) + 6c\right]$

$= \frac{w}{3}(2ax_2^2 + 2ax_0x_2 + 2ax_0^2 + 3bx_2 + 3bx_0 + 6c)$

$= \frac{w}{3}\left[(ax_2^2 + bx_2 + c) + (ax_0 + bx_0 + c) + a(x_2^2 + 2x_2x_0 + x_0^2) + 2b(x_2 + x_0) + 4c\right]$

$= \frac{w}{3}\left[y_2 + y_0 + a(x_2 + x_0)^2 + 2b(x_2 + x_0) + 4c\right]$

Now x_1 is halfway between x_0 and x_2 so $x_1 = \frac{x_0 + x_2}{2}$ or $x_2 + x_0 = 2x$,

So LHS $= \frac{w}{3}\left[y_2 + y_0 + a(2x_1)^2 + 2b(2x_1) + 4c\right]$

$= \frac{w}{3}\left[y_2 + y_0 + 4(ax_1^2 + bx_1 + c)\right]$

$= \frac{w}{3}(y_2 + y_0 + 4y_1)$

= RHS, as required.

Investigation — Simpson's rule and cubic functions

Show that $\int_a^b x^3\,dx = \frac{b-a}{4}(b+a)(b^2+a^2)$

$\int_a^b x^3\,dx = \left[\frac{1}{4}x^4\right]_a^b$

$= \frac{1}{4}\left[x^4\right]_a^b$

$= \frac{1}{4}(b^4 - a^4)$

$= \frac{1}{4}(b^2 - a^2)(b^2 + a^2)$

$= \frac{1}{4}(b-a)(b+a)(b^2+a^2)$

$= \frac{b-a}{4}(b+a)(b^2+a^2)$, as required

$w = \frac{b-a}{n}$ where $n = 2$

So $w = \frac{b-a}{2}$

n	x_n	y_n
0	$0 + a = a$	a^3
1	$a + \frac{b-a}{2} = \frac{b+a}{2}$	$\left(\frac{b+a}{2}\right)^3$
2	$a + b - a = b$	b^3

By Simpson's rule,

Area ≈ $\frac{w}{3}[y_b + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$

So $\int_a^b x^3\,dx \approx \frac{b-a}{6}\left[a^3+b^3+4\left(\frac{b+a}{2}\right)^3\right]$

$= \frac{b-a}{6}\left[a^3+b^3+\frac{4}{8}(b^3+3b^2a+3ba^2+a^3)\right]$

$= \frac{(b-a)}{2\times 6}\left[2a^3+2b^3+b^3+3b^2a+3ba^2+a^3\right]$

$= \frac{b-a}{12}\left[3a^3+3b^3+3b^2a+3ba^2\right]$

$= \frac{3(b-a)}{12}\left[b^3+b^2a+a^3+ba^2\right]$

$= \frac{b-a}{4}\left[b^2(b+a)+a^2(b+a)\right]$

$= \frac{b-a}{4}(b+a)(b^2+a^2)$

Which is the exact result obtained previously. So Simpson's rule gives the exact value for the area bounded by $y = x^3$.
Using a quadratic regression on a graphics calculator through the points (2, 8), (3, 27) and (4, 64) gives the equation $y = 9x^2 - 26x + 24$.
The two graphs are almost identical, but not quite. The parabola passes slightly underneath the cubic for $2 < x < 3$ and slightly above for $3 < x < 4$. So for $2 \le x \le 3$, the area under the parabola is slightly less than the area under the cubic, and for $3 \le x \le 4$ the area under the parabola is slightly greater than that of the cubic. However, these two slight differences cancel each other out when the area under both curves for $2 \le x \le 4$ is found, giving the exact same area under both curves for this region.

Chapter review

1 a Find an approximation for $f(x) = (2 + x)^4$ for small values of x

$f(x_0 + h) \approx f(x_0) + hf'(x_0)$

Let $f(x) = (2 + x)^4$

So $f'(x) = 4(2 + x)^3$

Let $x_0 = 0$ and $h = x$

$f(x) \approx f(0) + x f'(0)$

$(2 + x)^4 \approx (2)^4 + x \times 4(2)^3$

$(2 + x)^4 \approx 16 + 32x$

b $2.03^4 = (2 + 0.03)^4$

$2.03^4 \approx 16 + 32 \times 0.03$

$2.03^4 \approx 16.96$

c Percentage error $= \frac{|2.03^4 - 16.96|}{2.03^4} \times 100\%$

$= 0.13\%$

2 Estimate value of $\sqrt{50}$

$f(x_0 + h) \approx f(x_0) + hf'(x_0)$

Let $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

So $f'(x) = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}}$

Let $x_0 = 49$ and $h = 1$

$f(49 + 1) \approx f(49) + 1 \times f'(49)$

$\sqrt{50} \approx \sqrt{49} + \frac{1}{2\sqrt{49}}$

$\sqrt{50} \approx 7 + \frac{1}{14}$

$\sqrt{50} \approx 7\frac{1}{14}$

3 Find an approximate value for $3\sqrt{67}$

Let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

So $f'(x) = \frac{1}{3}x^{\frac{-2}{3}} = \frac{1}{3(\sqrt[3]{x})^2}$

$f(x_0 + h) \approx f(x_0) + hf'(x_0)$

Let $x_0 = 64$ and $h = 3$

$f(64 + 3) \approx \sqrt[3]{64} + 3 \times \frac{1}{3(\sqrt[3]{64})^2}$

$f(67) \approx 4 + \frac{1}{(4)^2}$

$f(67) \approx 4\frac{1}{16}$

4 $\int (x-1)(x^2-2x)^5\,dx$

Let $u = x^2 - 2x$

$\frac{du}{dx} = 2x - 2$ or $dx = \frac{du}{2x-2}$

So $\int (x-1)(x^2-2x)^5\,dx$

$= \int (x-1)(u)^5 \frac{du}{2x-2}$

$= \int (x-1)u^5 \frac{du}{2(x-1)}$

$= \frac{1}{2}\int u^5\,du$

$\therefore$ **B**

5 $\int \frac{\sin(x)}{\cos^3(x)}dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

So $\int \frac{\sin(x)}{\cos^3(x)}dx$

$= \int \frac{\sin(x)}{u^3} \times \frac{du}{-\sin(x)}$

$= -\int u^{-3}\,du$

$= -\left(\frac{-1}{2}u^{-2}\right)$

$= \frac{1}{2u^2}$

$= \frac{1}{2\cos^2(x)}$

6 $\int 6\sec^2(3x)\tan^4(3x)\,dx$

Let $u = \tan(3x)$

$\frac{du}{dx} = 3\sec^2(3x)$ or $dx = \frac{du}{3\sec^2(3x)}$

So $\int 6\sec^2(3x)\tan^4(3x)\,dx$

$= \int 6\sec^2(3x)u^4 \frac{du}{3\sec^2(3x)}$

$= 2\int u^4\,du$

$\therefore$ **A**

7 a $\int (\cos(x))e^{\sin(x)}\,dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

So $\int (\cos(x))e^{\sin(x)}\,dx$

$= \int (\cos(x))e^u \frac{du}{\cos(x)}$

$= \int e^u \, du$
$= e^u + c$
$= e^{\sin(x)} + c$

b $\int \frac{(\log_e(x))^2}{x} dx$

Let $u = \log_e(x)$

$\frac{du}{dx} = \frac{1}{x}$ or $dx = x \, du$

So $\int \frac{(\log_e(x))^2}{x} dx$

$= \int \frac{(u)^2}{x} x \, du$

$= \int u^2 \, du$

$= \frac{1}{3}u^3 + c$

$= \frac{1}{3}(\log_e(x))^3 + c$

8 $\int x(x+2)^{10} dx$

Let $u = x + 2$ and $x = u - 2$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int x(x+2)^{10} dx$

$= \int (u-2)(u)^{10} du$

$= \int (u^{11} - 2u^{10}) du$

$= \frac{1}{2}u^{12} - \frac{2}{11}u^{11} + c$

$= \frac{(x+2)^{12}}{12} - \frac{2(x+2)^{11}}{11} + c$

$= \frac{11(x+2)^{12} - 24(x+2)^{11}}{132} + c$

$= \frac{(x+2)^{11}[11(x+2) - 24]}{132} + c$

$= \frac{(x+2)^{11}(11x + 22 - 24)}{132} + c$

$= \frac{(x+2)^{11}(11x - 2)}{132} + c$

9 $\int x\sqrt{2-x} \, dx$

Let $u = 2 - x \, dx$ and $x = 2 - u$

$\frac{du}{dx} = -1$ or $dx = -du$

So $\int x\sqrt{2-x} \, dx$

$= \int (2-u)\sqrt{u} \times -du$

$= \int (u-2)u^{\frac{1}{2}} du$

$= \int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + c$

$= \frac{6u^{\frac{5}{2}} - 20u^{\frac{3}{2}}}{15} + c$

$= \frac{2u^{\frac{3}{2}}(3u - 10)}{15}$

$= \frac{2}{15}(2-x)^{\frac{3}{2}}[3(2-x) - 10]$

$= \frac{2}{15}(2-x)^{\frac{3}{2}}(6 - 3x - 10)$

$= \frac{-2}{15}(2-x)^{\frac{3}{2}}(3x + 4)$

10 $\int e^{2x}\sqrt{e^x - 1} \, dx$

Let $u = e^x - 1$ and $e^x = u + 1$

$\frac{du}{dx} = e^x$ or $dx = \frac{du}{e^x}$

So $\int e^{2x}\sqrt{e^x - 1} \, dx$

$= \int e^{2x}\sqrt{u} \, \frac{du}{e^x}$

$= \int e^x u^{\frac{1}{2}} du$

$= \int (u+1)u^{\frac{1}{2}} du$

$= \int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

$\therefore$ **A**

11 $\int \frac{x}{\sqrt{x+1}} dx$

Let $u = x + 1$ and $x = u - 1$

$\frac{du}{dx} = 1$ or $dx = du$

So $\int \frac{x}{\sqrt{x+1}} dx$

$= \int \frac{(u-1)}{\sqrt{u}} du$

$= \int \frac{(u-1)}{u^{\frac{1}{2}}} du$

$= \int \left(u^{\frac{1}{2}} - u^{\frac{-1}{2}}\right) du$

$= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + c$

$= \frac{2u^{\frac{3}{2}} - 6u^{\frac{1}{2}}}{3} + c$

$= \frac{2}{3}(u-3)u^{\frac{1}{2}} + c$

$= \frac{2}{3}(x+1-3)(x+1)^{\frac{1}{2}} + c$

$= \frac{2}{3}(x-2)\sqrt{x+1} + c$

12 $\int \cos^3(x)\sin^2(x) dx$

$= \int \cos(x)\cos^2(x)\sin^2(x) dx$

$= \int \cos(x)(1 - \sin^2(x))\sin^2(x) dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$ or $dx = \frac{du}{\cos(x)}$

So $\int \cos(x)(1 - \sin^2(x))\sin^2(x) dx$

$= \int \cos(x)(1 - u^2)u^2 \frac{du}{\cos(x)}$

$= \int (1-u^2)u^2\,du$

$= \int (u^2-u^4)\,du$

$\therefore$ **E**

13 If $f'(x) = 4\sin^2(x)$ and $f\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$, find $f(x)$

$f'(x) = 4\sin^2(x)$

$f(x) = \int 4\sin^2(x)\,dx$

$= \frac{4}{2}\int (1-\cos(2x))\,dx$

$= 2\int (1-\cos(2x))\,dx$

$= 2\left(x - \frac{1}{2}\sin(2x)\right)$

$= 2x - \sin(2x)$

$\therefore$ **E**

14 $\int \sin^5(x)\,dx$

$= \int \sin(x)(\sin^2(x))^2\,dx$

$= \int \sin(x)(1-\cos^2(x))^2\,dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

So $\int \sin(x)(1-\cos^2(x))^2\,dx$

$= \int \sin(x)(1-u^2)^2\frac{du}{-\sin(x)}$

$= -\int (1-u^2)^2\,du$

$= -\int (1-2u^2+u^4)\,du$

$= \int (2u^2-1-u^4)\,du$

$\therefore$ **C**

15 $\int (2+\tan^2(x))\,dx$

$= \int \left[1+(1+\tan^2(x))\right]dx$

$= \int (1+\sec^2(x))\,dx$

$= x + \tan(x) + c$

$\therefore$ **D**

16 **a** $\int \cos^2(2x)\,dx$

$= \frac{1}{2}\int (1+\cos(4x))\,dx$

$= \frac{1}{2}\left(x + \frac{1}{4}\sin(4x)\right) + c$

$= \frac{x}{2} + \frac{1}{8}\sin(4x) + c$

b $\int \sin^2\left(\frac{x}{4}\right)\cos^2\left(\frac{x}{4}\right)dx$

$= \frac{1}{4}\int \left(2\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)\right)^2 dx$

$= \frac{1}{4}\int \left(\sin\left(\frac{x}{2}\right)\right)^2 dx$

$= \frac{1}{4}\int \sin^2\left(\frac{x}{2}\right)dx$

$= \frac{1}{8}\int (1-\cos(x))\,dx$

$= \frac{1}{8}(x-\sin(x)) + c$

$= \frac{x-\sin(x)}{8} + c$

17 Find $f(x)$ if $f'(x) = \sin(2x)\cos(x)$ and $f(\pi) = 1$

$f'(x) = \sin(2x)$

$f(x) = \int \sin(2x)\cos(x)\,dx$

$= \int 2\sin(x)\cos(x)\cos(x)\,dx$

$= 2\int \sin(x)\cos^2(x)\,dx$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ or $dx = \frac{du}{-\sin(x)}$

So $f(x) = 2\int \sin(x)u^2\frac{du}{-\sin(x)}$

$= -2\int u^2\,du$

$= -2\left(\frac{1}{3}u^3\right) + c$

$f(x) = \frac{-2}{3}\cos^3(x) + c$

$f(\pi) = \frac{-2}{3}\cos^3(\pi) + c = 1$

$\frac{-2}{3}(-1)^3 + c = 1$

$\frac{2}{3} + c = 1$

$c = \frac{1}{3}$

So $f(x) = \frac{-2}{3}\cos^3(x) + \frac{1}{3}$

18 If $f'(x) = 2\sqrt{1-x^2}$ and $f(0) = -3$, find $f(x)$

$f'(x) = 2\sqrt{1-x^2}\,dx$

$f(x) = \int 2\sqrt{1-x^2}\,dx$

Let $x = \sin(\theta)$ and $\theta = \sin^{-1}(x)$

$\frac{dx}{d\theta} = \cos(\theta)$ or $dx = \cos(\theta)\,d\theta$

So $f(x) = \int 2\sqrt{1-\sin^2(\theta)}\cos(\theta)\,d\theta$

$= 2\int \cos(\theta)\cos(\theta)\,d\theta$

$= 2\int \cos^2(\theta)\,d\theta$

$= \frac{2}{2}\int (1+\cos(2\theta))\,d\theta$

$= \int (1+\cos(2\theta))\,d\theta$

$= \theta + \frac{1}{2}\sin(2\theta) + c$

$= \sin^{-1}(x) + \frac{1}{2}\sin(2\sin^{-1}(x)) + c$

$= \sin^{-1}(x) + \frac{2}{2}\sin(\sin^{-1}(x))\cos(\sin^{-1}(x)) + c$

$= \sin^{-1}(x) + x\cos(\sin^{-1}(x)) + c$

Now $\theta = \sin^{-1}(x)$

$x = \sin(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1$

$\cos^2(\theta) = 1 - \sin^2(\theta)$

$\cos(\theta) = \sqrt{1-\sin^2(\theta)}$

$\cos(\theta) = \sqrt{1-x^2}$

So $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$

Therefore $f(x) = \sin^{-1}(x) + x\sqrt{1-x^2} + c$

$f(0) = \sin^{-1}(0) + 0\sqrt{1-0^2} + c = -3$

$c = -3$

Therefore $f(x) = \sin^{-1}(x) + x\sqrt{1-x^2} - 3$

19 $\int \frac{1}{x^2-9x+20}\,dx$

$= \int\left(\frac{1}{x-5} - \frac{1}{x-4}\right)dx$

$= \log_e(x-5) - \log_e(x-4)$

$= \log_e\left(\frac{x-5}{x-4}\right)$

∴ **A**

20 $\int \frac{2x+3}{(x+1)^2}\,dx$

$\frac{2x+3}{(x+1)^2} = \frac{a}{x+1} + \frac{b}{(x+1)^2}$

$= \frac{a(x+1)+b}{(x+1)^2}$

So $2x+3 = a(x+1)+b$

Let $x = -1$, $-2+3 = b$

$b = 1$

Let $x = 0$, $3 = a+1$

$a = 2$

So $\int \frac{2x+3}{(x+1)^2}\,dx$

$= \int\left[\frac{2}{x+1} + \frac{1}{(x+1)^2}\right]dx$

$= \int\left[\frac{2}{x+1} + (x+1)^{-2}\right]dx$

$= 2\log_e(x+1) - (x+1)^{-1} + c$

$= 2\log_e(x+1) - \frac{1}{x+1} + c$

21 Find $\int f(x)\,dx$ where $f(x) = \frac{x^2-2x-12}{x^2-7x-8}$

$f(x) = \frac{x^2-2x-12}{x^2-7x-8}$

$= \frac{x^2-7x-8+5x-4}{x^2-7x-8}$

$= \frac{x^2-7x-8}{x^2-7x-8} + \frac{5x-4}{x^2-7x-8}$

$= 1 + \frac{5x-4}{(x+1)(x-8)}$

Now $\frac{5x-4}{(x+1)(x-8)} = \frac{a}{x+1} + \frac{b}{x-8}$

$= \frac{a(x-8)+b(x+1)}{(x+1)(x-8)}$

So $5x-4 = a(x-8) + b(x+1)$

Let $x = -1$, $-5-4 = -9a$

$-9 = -9a$

$a = 1$

Let $x = 8$, $40-4 = 9b$

$36 = 9b$

$b = 4$

So $\frac{5x-4}{(x+1)(x-8)} = \frac{1}{x+1} + \frac{4}{x-8}$

So $\int f(x)\,dx = \int\left(1 + \frac{1}{x+1} + \frac{4}{x-8}\right)dx$

$= x + \log_e(x+1) + 4\log_e(x-8) + c$

22 $\int x^2 e^x\,dx$

$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Let $u = x^2$ and $\frac{dv}{dx} = e^x$

So $\frac{du}{dx} = 2x$ and $v = e^x$

So $\int x^2 e^x\,dx$

$= x^2e^x - \int e^x(2x)\,dx$

$= x^2e^x - 2\int xe^x\,dx$

Let $u = x$ and $\frac{dv}{dx} = e^x$

So $\frac{du}{dx} = 1$ and $v = e^x$

So $\int x^2 e^x\,dx$

$= x^2e^x - 2(xe^x - \int e^x\,dx)$

$= x^2e^x - 2xe^x + 2\int e^x\,dx$

$= x^2e^x - 2xe^x + 2e^x + c$

$= e^x(x^2 - 2x + 2) + c$

23 $\int e^{2x}\sin(x)\,dx$

$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Let $u = \sin(x)$ and $\frac{dv}{dx} = e^{2x}$

So $\frac{du}{dx} = \cos(x)$ and $v = \frac{1}{2}e^{2x}$

So $\int e^{2x}\sin(x)\,dx$

$= \sin(x)\times\frac{1}{2}e^{2x} - \int\frac{1}{2}e^{2x}\cos(x)\,dx$

$= \frac{1}{2}e^{2x}\sin(x) - \frac{1}{2}\int e^{2x}\cos(x)\,dx$

Let $u = \cos(x)$ and $\frac{dv}{dx} = e^{2x}$

So $\frac{du}{dx} = -\sin(x)$ and $v = \frac{1}{2}e^{2x}$

So $\int e^{2x}\sin(x)\,dx$

$= \frac{1}{2}e^{2x}\sin(x) - \frac{1}{2}\left[\cos(x)\times\frac{1}{2}e^{2x} - \int\frac{1}{2}e^{2x}(-\sin(x))\,dx\right]$

$\int e^{2x}\sin(x)\,dx$

$= \frac{1}{2}e^{2x}\sin(x) - \frac{1}{4}e^{2x}\cos(x) - \frac{1}{4}\int e^{2x}\sin(x)\,dx$

$\frac{5}{4}\int e^{2x}\sin(x)\,dx = \frac{1}{4}e^{2x}(2\sin(x) - \cos(x))$

$\int e^{2x}\sin(x)\,dx = \frac{1}{5}e^{2x}(2\sin(x) - \cos(x)) + c$

24 For the integral to exist $9 - x^2 > 0$

$x^2 < 9$

$-3 < x < 3$

The largest domain it can be evaluated over is (–3, 3)
$\therefore$ **E**

25 a $\int_{-1}^{1} \frac{x^2}{x^3+1} dx$

For the integral to exist $x^3 + 1 \neq 0$

$x^3 \neq -1$

$x \neq -1$

The domain is R\{–1}, so the integral cannot be calculated.

b $\int_0^{\pi} 2x\cos(x)^2\, dx$

Let $u = x^2$

$\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

When $x = 0$, $u = 0^2 = 0$

When $x = \pi$, $u = \pi^2$

So $\int_0^{\pi} 2x\cos(x)^2\, dx$

$= \int_0^{\pi^2} 2x\cos(u)\frac{du}{2x}$

$= \int_0^{\pi^2} \cos(u)\, du$

$= [\sin(u)]_0^{\pi^2}$

$= \sin(\pi^2) - \sin(0)$

$= \sin(\pi^2)$

26 a $\int_0^2 \frac{1}{4+x^2} dx$

$= \frac{1}{2}\int_0^2 \frac{2}{4+x^2} dx$

$= \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$

$= \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$

$= \frac{1}{2}\left(\frac{\pi}{4}\right)$

$= \frac{\pi}{8}$

b $\int_{-2}^{1} \frac{x}{\sqrt{2-x}} dx$

Let $u = 2 - x$ and $x = 2 - u$

$\frac{du}{dx} = -1$ or $dx = -du$

When $x = -2$, $u = 2 - (-2) = 4$

When $x = 1$, $u = 2 - 1 = 1$

So $\int_{-2}^{1} \frac{x}{\sqrt{2-x}} dx$

$= \int_4^1 \frac{(2-u)}{\sqrt{u}} \times -du$

$= \int_4^1 \frac{u-2}{u^{\frac{1}{2}}} du$

$= \int_4^1 \left(u^{\frac{1}{2}} - 2u^{\frac{-1}{2}}\right) du$

$= \left[\frac{2}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right]_4^1$

$= \left[\frac{2}{3}(1)^{\frac{3}{2}} - 4(1)^{\frac{1}{2}}\right] - \left[\frac{2}{3}(4)^{\frac{3}{2}} - 4(4)^{\frac{1}{2}}\right]$

$= \left[\frac{2}{3} - 4\right] - \left[\frac{2\times 2^3}{3} - 4\times 2\right]$

$= \frac{2}{3} - 4 - \frac{16}{3} + 8$

$= 4 - \frac{14}{3}$

$= \frac{12-14}{3}$

$= \frac{-2}{3}$

27 $y = 2 - x^2 \quad y = \sqrt{x}$

$V = \pi\int_0^1 \left[(2-x^2)^2 - (\sqrt{x})^2\right] dx$

$= \pi\int_0^1 (4 - 4x^2 + x^4 - x) dx$

$\therefore$ **D**

28 $y = 2 - x^2 \qquad y = \sqrt{x}$

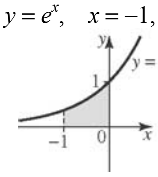
$x^2 = 2 - y \qquad x = y^2$

$V = \pi\int_1^2 (2-y)dy + \pi\int_0^1 (y^2)^2 dy$

$= \pi\int_1^2 (2-y)dy + \pi\int_0^1 y^4 dy$

$\therefore$ **D**

29 $y = e^x, \quad x = -1, \quad x = 0$



$V = \pi\int_{-1}^0 (e^x)^2 dx$

$= \pi\int_{-1}^0 e^{2x} dx$

$= \pi\left[\frac{1}{2}e^{2x}\right]_{-1}^0$

$= \frac{\pi}{2}\left[e^{2x}\right]_{-1}^0$

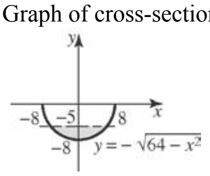
$= \frac{\pi}{2}\left[e^0 - e^{-2}\right]$

$= (1 - e^{-2})\frac{\pi}{2}$ cubic units

30 radius = 8 cm

depth = 3 cm

Graph of cross-section of bowl:



Let $y = -\sqrt{64-x^2}$, the lower semicircle of a circle of radius 8

To find the volume of water, rotate about the y-axis

$y = -\sqrt{64-x^2}$

$y^2 = 64 - x^2$

$x^2 = 64 - y^2$

$V = \pi\int_{-8}^{-5} (64 - y^2) dy$

$= \pi\left[64y - \frac{1}{3}y^3\right]_{-8}^{-5}$

$= \pi\left[\left(64\times(-5) - \frac{(-5)^3}{3}\right) - \left(64\times(-8) - \frac{(-8)^3}{3}\right)\right]$

$= \pi\left[64(-5+8) + \frac{1}{3}(5^3 - 8^3)\right]$

$= \pi\left[64\times 3 + \frac{1}{3}(125-512)\right]$

$= \pi\left[192 - \frac{1}{3}(387)\right]$

$= \pi\left[\frac{576-387}{3}\right]$

$= \frac{189}{3}\pi$

$= 63\pi\,\text{cm}^3$

31 $\int_0^{\pi} \frac{\sin(x)}{x}\,dx$, 4 strips

Answers to this question will vary as the answer is an approximation.

$w = \frac{b-a}{n}$ where $a = 0$, $b = \pi$ and $n = 4$

So $w = \frac{\pi - 0}{4}$

$= \frac{\pi}{4}$

n	x_n	y_n
0	0	1
1	$\frac{\pi}{4}$	0.900
2	$\frac{2\pi}{4}$	0.637
3	$\frac{3\pi}{4}$	0.300
4	$\frac{4\pi}{4}$	0

By Simpson's rule,

Area $\approx \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$

So $\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{w}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)]$

$= \frac{\pi}{12}[1 + 0 + 4(0.900 + 0.300) + 2(0.637)]$

$= 1.85$

Note: y_0 was taken to be 1 since $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, even though the point at $x = 0$ does not exist.

32 a i $\int_0^6 \sqrt{36 - x^2}\,dx$

Let $x = 6\sin(\theta)$

$\frac{dx}{d\theta} = 6\cos(\theta)$ or $dx = 6\cos(\theta)\,d\theta$

When $x = 0$, $0 = 6\sin(\theta)$

$\sin(\theta) = 0$

$\theta = 0$

When $x = 6$, $6 = 6\sin(\theta)$

$\sin(\theta) = 1$

$\theta = \frac{\pi}{2}$

So $\int_0^6 \sqrt{36-x^2}\,dx$

$= \int_0^{\frac{\pi}{2}} \sqrt{36-(6\sin(\theta))^2} \times 6\cos(\theta)\,d\theta$

$= 6\int_0^{\frac{\pi}{2}} \sqrt{36(1-\sin^2(\theta))}\cos(\theta)\,d\theta$

$= 36\int_0^{\frac{\pi}{2}} \cos(\theta)\cos(\theta)\,d\theta$

$= 36\int_0^{\frac{\pi}{2}} \cos^2(\theta)\,d\theta$

$= \frac{36}{2}\int_0^{\frac{\pi}{2}} (1+\cos(2\theta))\,d\theta$

$= 18\int_0^{\frac{\pi}{2}} (1+\cos(2\theta))\,d\theta$

$= 18\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$

$= 18\left[\left(\frac{\pi}{2}+\frac{1}{2}\sin(\pi)\right)-\left(0-\frac{1}{2}\sin(0)\right)\right]$

$= 18\left(\frac{\pi}{2}\right)$

$= 9\pi$

≈ 28.27

ii $\int_0^6 \sqrt{36-x^2}\,dx$, 6 strips

$w = \frac{b-a}{n}$ where $a = 0$, $b = 6$ and $n = 6$

So $w = \frac{6-0}{6}$

$= 1$

n	x_n	y_n
0	0	6
1	1	5.916
2	2	5.657
3	3	5.196
4	4	4.472
5	5	3.317
6	6	0

By Simpson's rule,

Area $\approx \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + ...) + 2(y_2 + y_4 + ...)]$

So $\int_0^6 \sqrt{36-x^2}\,dx \approx \frac{w}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

$= \frac{1}{3}[6 + 0 + 4(5.916 + 5.196 + 3.317) + 2(5.657 + 4.472)]$

$= 27.99$

b Percentage error $= \frac{|9\pi - 27.99|}{9\pi} \times 100\%$

$= 1\%$

There is approximately a 1% error when using Simpson's rule.

Modelling and problem solving

1 $P = I^2R$, $R = 600$ ohms

Let $f(I) = I^2 \times 600$

$= 600I^2$

So $f'(I) = 1200\,I$

Let $P_1 = 600\,I^2 = f(I)$ be the initial power and $P_2 = 600(I + \Delta I)^2 = f(I + \Delta I)$ be the final power so $P_2 - P_1 = \Delta P$ is the change in power

$f(x_0 + h) \approx f(x_0) + hf'(x_0)$

Let $x_0 = I$ and $h = \Delta I$

$f(I + \Delta I) \approx f(I) + \Delta I f'(I)$

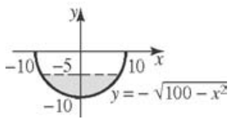
$P_2 \approx P_1 + \Delta I \times 1200I$

$P_2 - P_1 \approx 1200I\Delta I$

$\Delta P \approx 1200I\Delta I$

2 radius = 10 cm, depth = 5 cm

Graph of cross section of bowl and water



Let $y = -\sqrt{100 - x^2}$, equation of a semicircle which is the lower half of a circle of radius 10, centred at the origin.

To find the volume of water, rotate about y-axis

$y = -\sqrt{100 - x^2}$

$y^2 = 100 - x^2$

$x^2 = 100 - y^2$

$$V = \pi\int_{-10}^{-5}(100 - y^2)\,dy$$

$$= \pi\left[100y - \frac{1}{3}y^3\right]_{-10}^{-5}$$

$$= \pi\left[\left(100\times(-5) - \frac{(-5)^3}{3}\right) - \left(100\times(-10) - \frac{(-10)^3}{3}\right)\right]$$

$$= \pi\left[-500 + \frac{125}{3} + 1000 - \frac{1000}{3}\right]$$

$$= \pi\left[500 - \frac{875}{3}\right]$$

$$= \pi\left[\frac{1500 - 875}{3}\right]$$

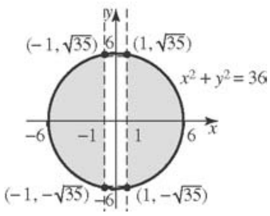
$$= \frac{625\pi}{3}\text{cm}^3$$

The volume of water is $\frac{625\pi}{3}\text{cm}^3$

3 Sphere: radius = 6 cm

Cylinder: radius = 1 cm, height = 6 cm

Graph of cross-section of object:



Let $x^2 + y^2 = 36$, circle of radius 6

$x = -1$ and $x = 1$

$x = \pm 1$, $(\pm 1)^2 + y^2 = 36$

$y^2 = 35$

$y = \pm\sqrt{35}$

The points of intersection are $(1, \sqrt{35}), (1, -\sqrt{35}), (-1, \sqrt{35}), (-1, -\sqrt{35})$

Find the volume of the object by rotating about the y-axis

$x^2 + y^2 = 36$

$x^2 = 36 - y^2$

$$V = \pi\int_{-\sqrt{35}}^{\sqrt{35}}\left[(36 - y^2) - 1^2\right]dy$$

$$= \pi\int_{-\sqrt{35}}^{\sqrt{35}}(35 - y^2)\,dy$$

$$= \pi\left[35y - \frac{1}{3}y^3\right]_{-\sqrt{35}}^{\sqrt{35}}$$

$= \pi\left[\left(35\sqrt{35} - \frac{(\sqrt{35})^3}{3}\right) - \left(35(-\sqrt{35}) - \frac{(-\sqrt{35})^3}{3}\right)\right]$

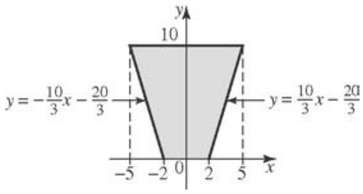
$= \pi\left[35\sqrt{35} - \frac{35\sqrt{35}}{3} + 35\sqrt{35} - \frac{35\sqrt{35}}{3}\right]$

$= \sqrt{35}\pi\left[70 - \frac{70}{3}\right]$

$= \sqrt{35}\pi\left[\frac{210-70}{3}\right]$

$= \frac{140\sqrt{35}\pi}{3}$ cm^3

4 Truncated cone: height = 10 cm
base radius = 5 cm
top radius = 2 cm
Graph of cross-section of object



So one line goes through (–5, 10) and (–2, 0), and the other through (2, 0) and (5, 10), giving lines with equations:
(–5, 10), (–2, 0)   (2, 0), (5, 10)

$m = \frac{0-10}{-2+5}$ $m = \frac{10-0}{5-2}$

$= \frac{-10}{3}$ $= \frac{10}{3}$

$y - y_1 = m(x - x_1)$ $y - y_1 = m(x - x_1)$

$y - 0 = \frac{-10}{3}(x+2)$ $y - 0 = \frac{10}{3}(x-2)$

$y = \frac{-10}{3}x - \frac{20}{3}$ $y = \frac{10}{3}x - \frac{20}{3}$

To find the volume of the cone, rotate about the y-axis

$y = \frac{10}{3}x - \frac{20}{3}$

$\frac{10x}{3} = y + \frac{20}{3}$

$x = \frac{3}{10}y + 2$

$V = \pi\int_0^{10}\left(\frac{3}{10}y + 2\right)^2 dy$

$= \pi\int_0^{10}\left(\frac{9}{100}y^2 + \frac{12}{10}y + 4\right)dy$

$= \pi\int_0^{10}\left(\frac{9}{100}y^2 + \frac{6}{5}y + 4\right)dy$

$= \pi\left[\frac{3}{100}y^3 + \frac{3}{5}y^2 + 4y\right]_0^{10}$

$= \pi\left[\left(\frac{3(10)^3}{100} + \frac{3(10)^2}{5} + 4(10)\right) - \left(\frac{3(0)^3}{100} + \frac{3(0)^2}{5} + 4(0)\right)\right]$

$= \pi\left[30 + \frac{300}{5} + 40\right]$

$= \pi[70 + 60]$

$= 130\pi$ cm^3

5 a radius = 4
Centre at (0, 6 + 4) → (0, 10)
$\Rightarrow (x-0)^2 + (y-10)^2 = 4^2$
$x^2 + (y-10)^2 = 16$

b Treat the upper and lower semicircles separately

$x^2 + (y - 10)^2 = 16$

$(y - 10)^2 = 16 - x^2$

$y - 10 = \pm\sqrt{16 - x^2}$

$y = 10 \pm \sqrt{16 - x^2}$

Upper semicircle: $y = 10 + \sqrt{16 - x^2}$

Lower semicircle: $y = 10 - \sqrt{16 - x^2}$

$$V = \pi\int_{-4}^{4}\left[(10+\sqrt{16-x^2})^2 - (10-\sqrt{16-x^2})^2\right]dx$$

$$= \pi\int_{-4}^{4}\left[(10+\sqrt{16-x^2}) + (10-\sqrt{16-x^2})\right]\left[(10+\sqrt{16-x^2}) - (10-\sqrt{16-x^2})\right]dx$$

$$= \pi\int_{-4}^{4} 20(2\sqrt{16-x^2})\,dx$$

$$= 40\pi\int_{-4}^{4}\sqrt{16-x^2}\,dx$$

Let $x = 4\sin(\theta)$

$\frac{dx}{d\theta} = 4\cos(\theta)$ or $dx = 4\cos(\theta)\,d\theta$

When $x = -4$, $-4 = 4\sin(\theta)$

$\sin(\theta) = -1$

$\theta = \frac{-\pi}{2}$

When $x = 4$, $4 = 4\sin(\theta)$

$\sin(\theta) = 1$

$\theta = \frac{\pi}{2}$

So $V = 40\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sqrt{16-(4\sin(\theta))^2}\times 4\cos(\theta)\,d\theta$

$$= 160\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sqrt{16(1-\sin^2(\theta))}\cos(\theta)\,d\theta$$

$$= 640\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\cos(\theta)\cos(\theta)\,d\theta$$

$$= 640\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\cos^2(\theta)\,d\theta$$

$$= \frac{640\pi}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(1+\cos(2\theta))\,d\theta$$

$$= 320\pi\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$= 320\pi\left[\left(\frac{\pi}{2}+\frac{1}{2}\sin(\pi)\right) - \left(\frac{-\pi}{2}+\frac{1}{2}\sin(-\pi)\right)\right]$$

$$= 320\pi\left[\frac{\pi}{2}+\frac{\pi}{2}\right]$$

$= 320\pi^2\ \text{cm}^3$

6 a Show $c = 2$ and find height of the vessel correct to 2 decimal places.

$y = 2\log_e(x), \quad 0 < x < e$

$y = x^2 - 2ex + e^2 + c, \quad e \le x \le 5$

The curve must be continuous at $x = e$ so

$2\log_e(e) = (e)^2 - 2e(e) + e^2 + c$

$2 = e^2 - 2e^2 + e^2 + c$

$2 = c$

$c = 2$, as required

The height of the vessel is the y-coordinate when $x = 5$

$y = (5)^2 - 2e(5) + e^2 + 2$

$= 25 - 10e + e^2 + 2$

$= 27 - 10e + e^2$

≈ 7.21 cm

b depth = 2 cm

$y = 2,\quad 2 = x^2 - 2ex + e^2 + 2$

$0 = x^2 - 2ex + e^2$

$0 = (x - e)^2$

$x = e$

So only the part modelled by $y = 2\log_e(x)$ needs to be rotated about the y-axis.

$$y = 2\log_e(x)$$

$$\frac{y}{2} = \log_e(x)$$

$$x = e^{\frac{y}{2}}$$

$$V = \pi\int_0^2 \left(e^{\frac{y}{2}}\right)^2 dy$$

$$= \pi\int_0^2 e^y\, dy$$

$$= \pi\left[e^y\right]_0^2$$

$$= \pi(e^2 - e^0)$$

$$= \pi(e^2 - 1)\text{ cm}^3$$

$$\approx 20.07\text{ cm}^3$$

The volume of wine is the glass is $\pi(e^2 - 1)$ or 20.07 cm^3.

c height = 7.2 cm (to the nearest mm)

$y = e^{\frac{y}{2}}\quad y = x^2 - 2ex + e^2 + 2$

$y = (x - e)^2 + 2$

$(x - e)^2 = y - 2$

$x - e = \pm\sqrt{y-2}$

$x = e \pm \sqrt{y-2}$

$\Rightarrow x = e + \sqrt{y-2}$

$$V = \pi\int_0^2 \left(e^{\frac{y}{2}}\right)^2 dy + \pi\int_2^{7.2}(e + \sqrt{y-2})^2\, dy$$

$$= \pi\int_0^2 e^y\, dy + \pi\int_2^{7.2}(e^2 + 2e\sqrt{y-2} + y - 2)\, dy$$

$$= \pi\int_0^2 e^y\, dy + \pi\int_2^{7.2}\left[y + 2e(y-2)^{\frac{1}{2}} + (e^2 - 2)\right] dy$$

$$= \pi(e^2 - 1) + \pi\left[\frac{1}{2}y^2 + \frac{4e}{3}(y-2)^{\frac{3}{2}} + (e^2 - 2)y\right]_2^{7.2}$$

$$= \pi(e^2 - 1) + \pi\left[\left(\frac{7.2^2}{2} + \frac{4e}{3}(7.2-2)^{\frac{3}{2}} + (e^2 - 2)\times 7.2\right) - \left(\frac{2^2}{2} + \frac{4e}{3}(2-2)^{\frac{3}{2}} + (e^2 - 2)\times 2\right)\right]$$

$$= \pi(e^2 - 1) + \pi\left[\frac{7.2^2}{2} + \frac{4e}{3}(5.2)^{\frac{3}{2}} + 5.2(e^2 - 2) - 2\right]$$

$$= 318.27\text{ cm}^3$$

The maximum volume the wineglass can hold is 318.27 cm^3.

7 a Maximum depth occurs when $x = 0$

$$y = \frac{2}{\sqrt{36 - x^2}},\ -5.98 \le x \le 5.98$$

$$x = 0,\quad y = \frac{2}{\sqrt{36 - 0^2}}$$

$$= \frac{2}{\sqrt{36}}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

maximum depth $= 4.086 - \frac{1}{3}$

$= 3.75$ m

The maximum depth is 3.75 m.

b $\int_{-5.98}^{5.98} \frac{2}{\sqrt{36-x^2}}\,dx$

$= 2\int_{-5.98}^{5.98} \frac{1}{\sqrt{36-x^2}}\,dx$

$= 2\left[\sin^{-1}\left(\frac{x}{6}\right)\right]_{-5.98}^{5.98}$

$= 2\left[\sin^{-1}\left(\frac{5.98}{6}\right) - \sin^{-1}\left(\frac{-5.98}{6}\right)\right]$

$= 2\left[\sin^{-1}\left(\frac{5.98}{6}\right) + \sin^{-1}\left(\frac{5.98}{6}\right)\right]$

$= 4\sin^{-1}\left(\frac{5.98}{6}\right)$

$= 5.96$ m^2

c $V = \pi\int_{-5.98}^{5.98} \left(\frac{2}{\sqrt{36-x^2}}\right)^2 dx$

$= 4\pi\int_{-5.98}^{5.98} \frac{1}{36-x^2}\,dx$

$\frac{1}{36-x^2} = \frac{1}{(6+x)(6-x)}$

$= \frac{a}{6+x} + \frac{b}{6-x}$

$= \frac{a(6-x)+b(6+x)}{(6+x)(6-x)}$

So $1 = a(6-x) + b(6+x)$

Let $x = -6$, $1 = 12a$

$a = \frac{1}{12}$

Let $x = 6$, $1 = 12b$

$b = \frac{1}{12}$

So $V = 4\pi\int_{-5.98}^{5.98} \left(\frac{1}{12(6+x)} + \frac{1}{12(6-x)}\right)dx$

$= \frac{4\pi}{12}\int_{-5.98}^{5.98} \left(\frac{1}{6+x} + \frac{1}{6-x}\right)dx$

$= \frac{\pi}{3}\left[\log_e(6+x) - \log_e(6-x)\right]_{-5.98}^{5.98}$

$= \frac{\pi}{3}\left[\log_e\left(\frac{6+x}{6-x}\right)\right]_{-5.98}^{5.98}$

$= \frac{\pi}{3}\left[\log_e\left(\frac{11.98}{0.02}\right) - \log_e\left(\frac{0.02}{11.98}\right)\right]$

$= \frac{\pi}{3}\left[\log_e(599) + \log_e(599)\right]$

$= \frac{2\pi}{3}\log_e(599)$

$= 13.39$ m^3

d V = Cross-sectional area × length

$= 5.96 \times 20$, from part (**b**)

$= 119.20$ m^3

The volume of dirt to be removed is 119.20 m^3.