Chapter 8 — Techniques of integration

Exercise 8A — The fundamental theorem of integral calculus

1 a $\int_0^1 x^2\ dx = \left[\frac{x^3}{3}\right]_0^1$

$= \frac{1}{3}$

b $\int_0^3 x^3\ dx = \left[\frac{x^4}{4}\right]_0^3$

$= \frac{81}{4}$

$= 20\ \frac{1}{4}$

c $\int_3^4 \left(x^2 - 2x\right)dx = \left[\frac{x^3}{3} - x^2\right]_3^4$

$= \left(\frac{64}{3} - 16\right) - (9 - 9)$

$= \frac{64 - 48}{3}$

$= \frac{16}{3}$

$= 5\frac{1}{3}$

d $\int_2^6 \frac{1}{x^2}\ dx = \int_2^6 x^{-2}dx$

$= \left[-x^{-1}\right]_2^6$

$= \left[-\frac{1}{x}\right]_2^6$

$= \frac{-1}{6} - \frac{-1}{2}$

$= \frac{-1}{6} + \frac{1}{2}$

$= \frac{1}{3}$

e $\int_0^2 x^3 + 3x^2 - 2x\ dx = \left[\frac{x^4}{4} + x^3 - x^2\right]_0^2$

$= 4 + 8 - 4$

$= 8$

f $\int_1^4 \left(3x^{\frac{1}{2}} + 2x^2\right)dx = \left[2x^{\frac{3}{2}} + \frac{2}{3}x^3\right]_1^4$

$= \left(2 \times 8 + \frac{2}{3} \times 64\right) - \left(2 + \frac{2}{3}\right)$

$= \frac{176}{3} - \frac{8}{3}$

$= 56$

g $\int_{-1}^1 \left(6 - 2x + x^2\right)dx = \left[6x - x^2 + \frac{x^3}{3}\right]_{-1}^1$

$= \left(6 - 1 + \frac{1}{3}\right) - \left(-6 - 1 - \frac{1}{3}\right)$

$= \frac{16}{3} - \frac{-22}{3}$

$= \frac{38}{3}$

$= 12\frac{2}{3}$

h $\int_{-4}^{-2} \left(x^3 + x - 4\right)dx = \left[\frac{x^4}{4} + \frac{x^2}{2} - 4x\right]_{-4}^{-2}$

$= (4 + 2 + 8) - (64 + 8 + 16)$

$= 14 - 88$

$= -74$

i $\int_4^9 3\sqrt{x}\ dx = \int_4^9 3x^{\frac{1}{2}}dx$

$= \left[2x^{\frac{3}{2}}\right]_4^9$

$= (2 \times 27) - (2 \times 8)$

$= 54 - 16$

$= 38$

j $\int_1^2 \left(4x^{-2} + 2x - 6\right)dx = \left[-4x^{-1} + x^2 - 6x\right]_1^2$

$= \left[\frac{-4}{x} + x^2 - 6x\right]_1^2$

$= (-2 + 4 - 12) - (-4 + 1 - 6)$

$= -10 - {}^{-}9$

$= -1$

k $\int_0^3 2\left(x + 4\right)^4\ dx = 2\int_0^3 \left(x + 4\right)^4\ dx$

$= 2\left[\frac{(x+4)^5}{5}\right]_0^3$

$= 2\left[\left(\frac{7^5}{5}\right) - \left(\frac{4^5}{5}\right)\right]$

$= 6313{\cdot}2$

l $\int_{-1}^2 3(5x - 2)^4\ dx = 3\int_{-1}^2 (5x - 2)^4 dx$

$= 3\left[\frac{(5x-2)^5}{5 \times 5}\right]_{-1}^2$

$= 3\left[\left(\frac{8^5}{25}\right) - \left(\frac{-7^5}{25}\right)\right]$

$= 5949$

m $\int_{-2}^0 -4(2 - 3x)^3\ dx = -4\int_{-2}^0 (2 - 3x)^3\ dx$

$= -4\left[\frac{(2-3x)^4}{4 \times {}^{-}3}\right]_{-2}^0$

$= -4\left[\left(\frac{2^4}{-12}\right) - \left(\frac{8^4}{-12}\right)\right]$

$= -1360$

n $\int_0^3 \frac{5}{(2x-7)^3}dx = 5\int_0^3 (2x - 7)^{-3}\ dx$

$$= 5\left[\frac{(2x-7)^{-2}}{2\times{}^{-}2}\right]_0^3$$

$$= 5\left[\left(\frac{(-1)^{-2}}{-4}\right)-\left(\frac{(-7)^{-2}}{-4}\right)\right]$$

$$= 5\left[\frac{1}{-4}-\frac{1}{-196}\right]$$

$$= -\frac{60}{49}$$

o $\int_1^4\left(2x^{\frac{3}{2}}-3x^{-1}\right)dx$

$$= \left[\frac{4x^{\frac{5}{2}}}{5}-3\log_e x\right]_1^4$$

$$= \left(\frac{4\times 32}{5}-3\log_e 4\right)-\left(\frac{4}{5}-3\log_e 1\right)$$

$$= \frac{128}{5}-3\log_e 4-\frac{4}{5}$$

$$= \frac{124}{5}-3\log_e 4$$

$$= 24{\cdot}8-3\log_e 4$$

$$= 20{\cdot}64$$

p $\int_1^3\frac{2x^3+5x^2}{x}dx=\int_1^3 2x^2+5x\,dx$

$$= \left[\frac{2x^3}{3}+\frac{5x^2}{2}\right]_1^3$$

$$= \left(18+\frac{45}{2}\right)-\left(\frac{2}{3}+\frac{5}{2}\right)$$

$$= 17\frac{1}{3}+20$$

$$= 37\frac{1}{3}$$

q $\int_1^5\frac{3}{5x}dx=\frac{3}{5}\int_1^5\frac{1}{x}dx$

$$= \frac{3}{5}[\log_e x]_1^5$$

$$= \frac{3}{5}\log_e 5$$

$$\simeq 0{\cdot}966$$

r $\int_0^2\frac{-4}{(3x-4)^5}dx=-4\int_0^2(3x-4)^{-5}dx$

$$= -4\left[\frac{(3x-4)^{-4}}{3\times -4}\right]_0^2$$

$$= -4\left[\frac{2^{-4}}{-12}-\frac{(-4)^{-4}}{-12}\right]$$

$$= -4\left[\frac{1}{-16\times 12}+\frac{1}{256\times 12}\right]$$

$$= \frac{-5}{256}$$

s $\int_3^7\frac{1}{\sqrt{2x-5}}dx=\int_3^7(2x-5)^{-\frac{1}{2}}dx$

$$= \left[\frac{(2x-5)^{\frac{1}{2}}}{\frac{1}{2}\times 2}\right]_3^7$$

$$= \left[\sqrt{2x-5}\right]_3^7$$

$$= \sqrt{9}-\sqrt{1}$$

$$= 3-1$$

$$= 2$$

t $\int_{-2}^0\frac{6}{\sqrt{8-3x}}dx=6\int_{-2}^0(8-3x)^{\frac{-1}{2}}dx$

$$= 6\left[\frac{(8-3x)^{\frac{1}{2}}}{\frac{1}{2}\times{}^{-}3}\right]_{-2}^0$$

$$= 6\left[\frac{-2}{3}\sqrt{8-3x}\right]_{-2}^0$$

$$= 6\times 0{\cdot}6088$$

$$\simeq 3{\cdot}65$$

2 a $\int_0^2 e^{4x}dx=\left[\frac{1}{4}e^{4x}\right]_0^2$

$$= \frac{1}{4}e^8-\frac{1}{4}$$

$$= \frac{1}{4}\left(e^8-1\right)$$

b $\int_{-2}^0 e^{\frac{x}{3}}dx=\left[3e^{\frac{x}{3}}\right]_{-2}^0$

$$= 3e^0-3e^{\frac{-2}{3}}$$

$$= 3\left(1-e^{\frac{-2}{3}}\right)$$

c $\int_{-1}^1-4e^{-2x}dx=\left[2e^{-2x}\right]_{-1}^1$

$$= 2e^{-2}-2e^2$$

$$= 2e^{-2}\,2e^2$$

$$= 2(e^{-2}-e^2)$$

d $\int_1^2\left(3e^{6x}+5x\right)dx=\left[\frac{1}{2}e^{6x}+\frac{5x^2}{2}\right]_1^2$

$$= \left(\frac{1}{2}e^{12}+10\right)-\left(\frac{1}{2}e^6+\frac{5}{2}\right)$$

$$= \frac{1}{2}e^{12}-\frac{1}{2}e^6+\frac{15}{2}$$

$$= \frac{1}{2}\left(e^{12}+15-e^6\right)$$

e $\int_1^4\left(\frac{5}{x}+e^{\frac{x}{2}}\right)dx=\left[5\log_e x+2e^{\frac{x}{2}}\right]_1^4$

$$= (5\log_e 4+2e^2)-\left(5\log_e 1+2e^{\frac{1}{2}}\right)$$

$$= 5\log_e 4+2e^2-2e^{\frac{1}{2}}$$

f $\int_{-3}^{-1}\left(e^{2x}-e^{-2x}\right)dx=\left[\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}\right]_{-3}^{-1}$

$$= \left(\frac{1}{2}e^{-2}+\frac{1}{2}e^2\right)-\left(\frac{1}{2}e^{-6}+\frac{1}{2}e^6\right)$$

$= \frac{1}{2}(e^{-2} + e^{2} - e^{-6} - e^{6})$

g $\int_0^{\frac{\pi}{2}} \sin x \, dx = \left[-\cos x \right]_0^{\frac{\pi}{2}}$

$= \left(-\cos\frac{\pi}{2} \right) - \left(-\cos 0 \right)$

$= 0 + 1$

$= 1$

h $\int_{\frac{\pi}{2}}^{\pi} 3\sin 4x \, dx = \left[\frac{-3}{4}\cos 4x \right]_{\frac{\pi}{2}}^{\pi}$

$= \left(\frac{-3}{4}\cos 4\pi \right) - \left(\frac{-3}{4}\cos 2\pi \right)$

$= \frac{-3}{4} + \frac{3}{4}$

$= 0$

i $\int_0^{\pi} 5\sin\frac{x}{4} \, dx = \left[-20\cos\frac{x}{4} \right]_0^{\pi}$

$= \left(-20\cos\frac{\pi}{4} \right) - (-20\cos 0)$

$= -20 \times \frac{\sqrt{2}}{2} + 20$

$= 20 - 10\sqrt{2}$

j $\int_{\pi}^{2\pi} -2\sin\frac{x}{3} \, dx = \left[6\cos\frac{x}{3} \right]_{\pi}^{2\pi}$

$= \left(6\cos\frac{2\pi}{3} \right) - \left(6\cos\frac{\pi}{3} \right)$

$= -3 - 3$

$= -6$

k $\int_{-\pi}^{0} \cos 2x \, dx = \left[\frac{1}{2}\sin 2x \right]_{\pi}^{0}$

$= \left(\frac{1}{2}\sin 0 \right) - \left(\frac{1}{2}\sin(-2\pi) \right)$

$= 0 - 0$

$= 0$

l $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 8\cos 4x \, dx = \left[2\sin 4x \right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$

$= (2\sin 2\pi) - (2\sin(-2\pi))$

$= 0 - 0$

$= 0$

m $\int_{\pi}^{2\pi} 3\cos\frac{x}{6} \, dx = \left[18\sin\frac{x}{6} \right]_{\pi}^{2\pi}$

$= \left(18\sin\frac{\pi}{3} \right) - \left(18\sin\frac{\pi}{6} \right)$

$= 18 \times \frac{\sqrt{3}}{2} - 18 \times \frac{1}{2}$

$= 9\sqrt{3} - 9$

n $\int_{-2\pi}^{0} -7\cos\frac{x}{2} \, dx = \left[-14\sin\frac{x}{2} \right]_{-2\pi}^{0}$

$= (-14\sin 0) - (-14\sin(-\pi))$

$= 0 - 0$

$= 0$

o $\int_{-\pi}^{\frac{-\pi}{2}} -4\cos 3x \, dx = \left[\frac{-4}{3}\sin 3x \right]_{-\pi}^{\frac{-\pi}{2}}$

$= \left(\frac{-4}{3}\sin\frac{-3\pi}{2} \right) - \left(\frac{-4}{3}\sin(-3\pi) \right)$

$= -\frac{4}{3} - 0$

$= \frac{-4}{3}$

p $\int_0^{\pi} \pi\left(2 + \sin\frac{x}{4} \right) dx = \pi\int_0^{\pi}\left(2 + \sin\frac{x}{4} \right) dx$

$= \pi\left[2x - 4\cos\frac{x}{4} \right]_0^{\pi}$

$= \pi\left[\left(2\pi - 4\cos\frac{\pi}{4} \right) - (0 - 4\cos 0) \right]$

$= \pi\left(2\pi - 2\sqrt{2} + 4 \right)$

$\simeq 23{\cdot}4$

q $\int_1^3 \left(x^2 + 3 - 6\sin 3x \right) dx = \left[\frac{x^3}{3} + 3x + 2\cos 3x \right]_1^3$

$= (9 + 9 + 2\cos 9) - \left(\frac{1}{3} + 3 + 2\cos 3 \right)$

$= 14\frac{2}{3} + 2\cos 9 - 2\cos 3$

r $\int_1^{\pi} \left(\frac{1}{x} + 3\cos\frac{x}{2} \right) dx = \left[\log_e x + 6\sin\frac{x}{2} \right]_1^{\pi}$

$= \left(\log_e \pi + 6\sin\frac{\pi}{2} \right) - \left(\log_e 1 + 6\sin\frac{1}{2} \right)$

$= \log_e \pi + 6 - 6\sin\frac{1}{2}$

3 if $\int_1^5 f(x) \, dx = 6$

then $3\int_1^5 f(x) \, dx = 3 \times 6 = 18$

$\int 3f(x) \, dx = 18$

4 $\int_1^5 f(x) \, dx = 6$

a $\int_1^5 \left(f(x) + 1 \right) dx = \int_1^5 f(x) \, dx + \int_1^5 1 \, dx$

$= 6 + [x]_1^5$

$= 6 + (5 - 1)$

$= 10$

Answer is **B**.

b $\int_5^1 f(x) \, dx = -\int_1^5 f(x) \, dx$

$= -6$

Answer is **A**.

5 a $\int_0^3 \left(t^2 - 4t \right) dt = \left[\frac{t^3}{3} - 2t^2 \right]_0^3$

$= 9 - 18$

$= -9$

b $\int_0^{\frac{\pi}{2}} 2\cos 3t \, dt = \left[\frac{2}{3}\sin 3t \right]_0^{\frac{\pi}{2}}$

$= \left(\frac{2}{3}\sin\frac{3\pi}{2} \right) - \left(\frac{2}{3}\sin 0 \right)$

$= \frac{-2}{3} - 0$

$= \frac{-2}{3}$

c $\int_4^7 \frac{3}{(p-3)^{\frac{3}{2}}}\,dp = \int_4^7 3(p-3)^{\frac{-3}{2}}\,dp$

$$= 3\left[\frac{(p-3)^{\frac{-1}{2}}}{\frac{-1}{2}\times 1}\right]_4^7$$

$$= 3\left[-\frac{2}{\sqrt{p-3}}\right]_4^7$$

$$= 3\left[\left(\frac{-2}{2}\right)-\left(\frac{-2}{1}\right)\right]$$

$$= 3(-1+2)$$

$$= 3$$

d $\int_{5\pi}^{10\pi}\left(-\sin\frac{x}{5}\right)dx = \left[5\cos\frac{x}{5}\right]_{5\pi}^{10\pi}$

$$= (5\cos 2\pi) - (5\cos\pi)$$

$$= 5 + 5$$

$$= 10$$

e $\int_0^{\pi}\left(e^{\frac{x}{4}} - \cos 2x\right)dx = \left[4e^{\frac{x}{4}} - \frac{1}{2}\sin 2x\right]_0^{\pi}$

$$= \left(4e^{\frac{\pi}{4}} - \frac{1}{2}\sin 2\pi\right) - \left(4e^0 - \frac{1}{2}\sin 0\right)$$

$$= 4e^{\frac{\pi}{4}} - 4$$

f $\int_1^2 \frac{8}{4m-3}\,dm = 8\int_1^2 \frac{1}{4m-3}\,dm$

$$= \left[8\times\frac{1}{4}\log_e(4m-3)\right]_1^2$$

$$= 2\log_e 5 - 2\log_e 1$$

$$= 2\log_e 5$$

g $\int_1^4 \frac{-3}{t\sqrt{t}}\,dt = \int_1^4 -3t^{\frac{-3}{2}}\,dt$

$$= \left[\frac{-3t^{\frac{-1}{2}}}{\frac{-1}{2}}\right]_1^4 = \left[6t^{\frac{-1}{2}}\right]_1^4$$

$$= \left[\frac{6}{\sqrt{t}}\right]_1^4$$

$$= 3 - 6$$

$$= -3$$

h $\int_{-1}^{1}\left(3\sin 2x - e^{-3x}\right)dx$

$$= \left[-\frac{3}{2}\cos 2x + \frac{1}{3}e^{-3x}\right]_{-1}^{1}$$

$$= \left(-\frac{3}{2}\cos 2 + \frac{1}{3}e^{-3}\right) - \left(-\frac{3}{2}\cos(-2) + \frac{1}{3}e^{3}\right)$$

$$= -\frac{3}{2}\cos 2 + \frac{3}{2}\cos(-2) + \frac{1}{3}e^{-3} - \frac{1}{3}e^{3}$$

$$= \frac{1}{3}(e^{-3} - e^{3})$$

6 $\int_0^k (2x+3)\,dx = 4$

$$\left[x^3 + 3x\right]_0^k = 4$$

$$k^2 + 3k = 4$$

$$k^2 + 3k - 4 = 0$$

$$(k+4)(k-1) = 0$$

$$k = -4 \text{ or } 1$$

7 $\int_0^k 3x^3\,dx = 8$

$$\left[x^3\right]_0^k = 8$$

$$k^3 = 8$$

$$k = 2$$

8 $\int_1^k \frac{2}{x}\,dx = \log_e 9$

$$\left[2\log_e x\right]_1^k = \log_e 9$$

$$2\log_e k - 2\log_e 1 = \log_e 9$$

$$2\log_e k = \log_e 9$$

$$\log_e k^2 = \log_e 9$$

$$k^2 = 9$$

$$k = \pm 3$$

$k \neq -3$ as $\frac{2}{x}$ has an asymptote at $x = 0$

$\Rightarrow k = 3.$

9 $\int_0^a e^{\frac{x}{2}}\,dx = 4$

$$\left[2e^{\frac{x}{2}}\right]_0^a = 4$$

$$2e^{\frac{a}{2}} - 2e^0 = 4$$

$$2e^{\frac{a}{2}} - 2 = 4$$

$$2e^{\frac{a}{2}} = 6$$

$$e^{\frac{a}{2}} = 3$$

$$\frac{a}{2} = \log_e 3$$

$$a = 2\log_e 3$$

10 $\int_k^{\pi}\cos 2x\,dx = \frac{-\sqrt{3}}{4} \quad 0 < k < \frac{\pi}{2}$

$$\left[\frac{1}{2}\sin 2x\right]_k^{\pi} = \frac{-\sqrt{3}}{4}$$

$$\frac{1}{2}\sin 2\pi - \frac{1}{2}\sin 2k = \frac{-\sqrt{3}}{4}$$

$$-\frac{1}{2}\sin 2k = \frac{-\sqrt{3}}{4}$$

$$\sin 2k = \frac{\sqrt{3}}{2}$$

basic angle is $\frac{\pi}{3}$

1st and 2nd quadrant

$$2k = \frac{\pi}{3},\ \pi - \frac{\pi}{3}$$

$$= \frac{\pi}{3}, \frac{2\pi}{3}$$

$$k = \frac{\pi}{6},\ \frac{\pi}{3}$$

Exercise 8B — Signed areas

1 a Area $= \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times 4 \times 4$$

$$= 8 \text{ sq.units.}$$

b Area $= \int_0^4 x \ dx$

$= \left[\frac{x^2}{2}\right]_0^4$

$= 8$ sq.units.

2 a Area $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times 3 \times 3$

$= \frac{9}{2}$ sq. units

$= 4{\cdot}5$ sq. units.

b Area $= \int_0^3 3 - x\,dx$

$= \left[3x - \frac{x^2}{2}\right]_0^3$

$= \left(9 - \frac{9}{2}\right) - 0$

$= \frac{9}{2}$

$= 4{\cdot}5$ sq. units

3 a Area $= \int_1^3 2x\,dx$

b Area $= \int_0^4 (4 - x)\,dx$

c Area $= \int_1^2 x^2 dx$

d Area $= \int_{-3}^{-1} 3x^2 dx$

e Area $= \int_1^3 \left(x^3 - 9x^2 + 20x\right)dx$

f Area $= \int_{-2}^0 \left(-x^3 - 4x^2 - 4x\right)dx$

g Area $= \int_{-1}^1 e^x dx$

h Area $= \int_1^4 e^{-2x} dx$

i Area $= \int_0^{\frac{\pi}{2}} 2\sin 2x\,dx$

j Area $= \int_0^{\frac{3\pi}{2}} \cos\frac{x}{3} dx$

4 a $A = \int_1^3 2x\,dx$

$= \left[x^2\right]_1^3$

$= 9 - 1$

$= 8$ sq. units.

b $A = \int_0^4 (4 - x)\,dx$

$= \left[4x - \frac{x^2}{2}\right]_0^4$

$= (16 - 8) - 0$

$= 8$ sq. units.

c $A = \int_1^2 x^2\,dx$

$= \left[\frac{x^3}{3}\right]_1^2$

$= \frac{8}{3} - \frac{1}{3}$

$= \frac{7}{3}$ sq. units.

d $A = \int_{-3}^{-1} 3x^2\,dx$

$= \left[x^3\right]_{-3}^{-1}$

$= -1 - (-27)$

$= 26$ sq. units

e $A = \int_1^3 \left(x^3 - 9x^2 + 20x\right)dx$

$= \left[\frac{x^4}{4} - 3x^3 + 10x^2\right]_1^3$

$= \left(\frac{81}{4} - 81 + 90\right) - \left(\frac{1}{4} - 3 + 10\right)$

$= 22$ sq. units.

f $A = \int_{-2}^0 \left(-x^3 - 4x^2 - 4x\right)dx$

$= \left[\frac{-x^4}{4} - \frac{4x^3}{3} - 2x^2\right]_{-2}^0$

$= 0 - \left(-4 - \frac{-5}{3} - 8\right)$

$= 4 - \frac{-5}{3} + 8$

$= 1\frac{1}{3}$ sq. units.

g $A = \int_{-1}^1 e^x\,dx$

$= \left[e^x\right]_{-1}^1$

$= e - e^{-1}$ sq. units.

h $A = \int_1^4 e^{-2x}\,dx$

$= \left[\frac{1}{-2e} - 2x\right]_1^4$

$= \left[\frac{-1}{2}e^{-8} + \frac{1}{2}e^{-2}\right]$

$= \frac{-1}{2}(e^{-8} - e^{-2})$sq.units.

i $A = \int_0^{\frac{\pi}{2}} 2\sin 2x\,dx$

$= \left[-\cos 2x\right]_0^{\frac{\pi}{2}}$

$= -\cos\pi - -\cos 0$

$= 1 + 1$

$= 2$ sq. units.

j $A = \int_0^{3\frac{\pi}{2}} \cos\frac{x}{3} dx$

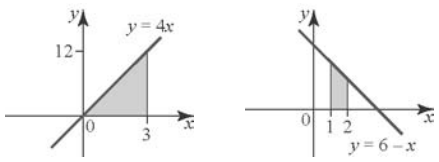
$= \left[3\sin\frac{x}{3}\right]_0^{\frac{3\pi}{2}}$

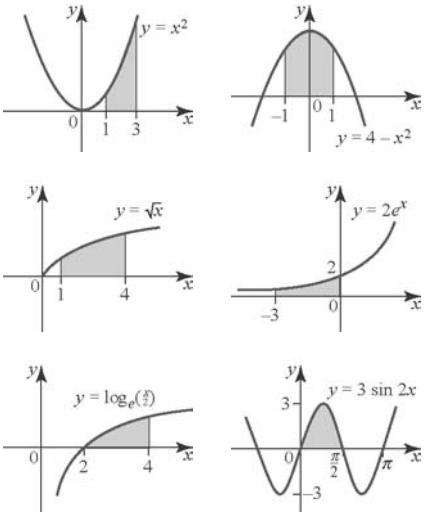
$= 3\sin\frac{\pi}{2} - 3\sin 0$

$= 3 - 0$

$= 3$ sq. units.

5 a–h





6 a $A = -\int_0^2 (x-2)\,dx$

$= -\left[\frac{x^2}{2} - 2x\right]_0^2$

$= -[(2-4)-0]$

$= -(-2)$

Area = 2 sq. units

b $A = -\int_{-2}^{-1} (-4-2x)\,dx$

$= -\left[-4x - x^2\right]_{-2}^{-1}$

$= -[(4-1)-(8-4)]$

$= -(3-4)$

$= -(-1)$

$= 1$ sq. unit

c $A = -\int_0^2 (x^2-4)\,dx$

$= -\left[\frac{x^3}{3} - 4x\right]_0^2$

$= -\left[\frac{8}{3} - 8\right]$

$= -\left(\frac{-16}{3}\right)$

$= 5\frac{1}{3}$ sq. units

d $A = -\int_{-2}^{-1} (1-x^2)\,dx$

$= -\left[x - \frac{x^3}{3}\right]_{-2}^{-1}$

$= -\left[\left(-1+\frac{1}{3}\right) - \left(-2+\frac{8}{3}\right)\right]$

$= -\left[\frac{-2}{3} - \frac{2}{3}\right]$

$= -\left(-\frac{4}{3}\right)$

$= \frac{4}{3}$ sq. units

e $A = -\int_{-2}^{0} x^3\,dx$

$= -\left[\frac{x^4}{4}\right]_{-2}^{0}$

$= -[0-4]$

$= -(-4)$

$= 4$ sq. units

f $A = -\int_{-1}^{1} (x^3 + 2x^2 - x - 2)\,dx$

$= -\left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{1}$

$= -\left[\left(\frac{1}{4} + \frac{2}{3} - \frac{1}{2} - 2\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right)\right]$

$= -\left(\frac{4}{3} - 4\right)$

$= -\left(\frac{-8}{3}\right)$

$= 2\frac{2}{3}$ sq. units

g $A = -\int_{-1}^{1} -e^x\,dx$

$= -\left[-e^x\right]_{-1}^{1}$

$= -[-e-(-e^{-1})]$

$= -[e^{-1} - e]$

$= e - e^{-1}$ sq. units

h $A = -\int_{\frac{1}{2}}^{1} -e^{-2x}\,dx$

$= -\left[\frac{1}{2}e^{-2x}\right]_{\frac{1}{2}}^{1}$

$= -\left[\left(\frac{1}{2}e^{-2}\right) - \left(\frac{1}{2}e^{-1}\right)\right]$

$= \frac{1}{2}(e^{-1} - e^{-2})$ sq. units

i $A = -\int_{\pi}^{\frac{3\pi}{2}} \sin x\,dx$

$= -\left[-\cos x\right]_{\pi}^{\frac{3\pi}{2}}$

$= -(-\cos\ \frac{3\pi}{2} - -\cos\ \pi)$

$= 0 - -1$

$= 1$ sq. unit

j $A = -\int_{-2\pi}^{-\pi} 2\cos\frac{x}{2}\,dx$

$= -\left[4\sin\frac{x}{2}\right]_{-2\pi}^{\pi}$

$= -[4\sin\ \frac{-\pi}{2} - 4\sin -\pi]$

$= -[4 \times -1 - 4 \times 0]$

$= -(-4)$

$= 4$ sq. units

7 a $A = -\int_{-2}^{-1} f(x)\ dx$

Answer is **E**

b $A = -\int_{-2}^{0} f(x)\,dx + \int_0^3 f(x)\,dx$

$= \int_0^{-2} f(x)\,dx + \int_0^3 f(x)\,dx$

$= \int_0^3 f(x)\,dx + \int_0^{-2} f(x)\,dx$

Answer is **A**.

8 a $A = \int_0^2 f(x)\mathrm{d}x + -\int_2^5 f(x)\mathrm{d}x$

$= \int_0^2 f(x)\mathrm{d}x - \int_2^5 f(x)\mathrm{d}x$

b $A = -\int_{-3}^1 g(x)\mathrm{d}x + \int_1^3 g(x)\mathrm{d}x$

$= \int_1^3 g(x)\mathrm{d}x - \int_{-3}^1 g(x)\mathrm{d}x$

c $A = -\int_{-3}^{-1} h(x)\mathrm{d}x + \int_{-1}^2 h(x)\mathrm{d}x$

$= \int_{-1}^2 h(x)\mathrm{d}x - \int_{-3}^{-1} h(x)\mathrm{d}x$

d $A = -\int_{-5}^{-4} f(x)\mathrm{d}x + \int_{-4}^{-2} f(x)\mathrm{d}x$

$= \int_{-4}^{-2} f(x)\mathrm{d}x - \int_{-5}^{-4} f(x)\mathrm{d}x$

e $A = \int_{-3}^{-2} g(x)\mathrm{d}x - \int_{-2}^2 g(x)\mathrm{d}x + \int_2^3 g(x)\mathrm{d}x$

$= \int_2^3 g(x)\mathrm{d}x - \int_{-2}^2 g(x)\mathrm{d}x + \int_{-3}^{-2} g(x)\mathrm{d}x$

9 a $A = \int_{-2}^1 x^3 - 2x^2 - 5x + 6$

$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right]_{-2}^1$

$= \left(\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6\right) - \left(4 + \frac{16}{3} - 10 - 12\right)$

$= 15\frac{3}{4}$ sq. units

Answer is **B**

b $A = -\int_1^3 (x^3 - 2x^2 - 5x + 6)\mathrm{d}x$

$= -\left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right]_1^3$

$= -\left[\left(\frac{81}{4} - 18 - \frac{45}{2} + 18\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6\right)\right]$

$= -\left(-5\frac{1}{3}\right)$

$= 5\frac{1}{3}$ sq. units

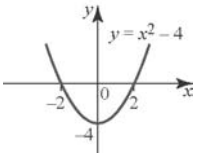
Answer is **D**.

c $A = 15\frac{3}{4} + 5\frac{1}{3}$

$= 21\frac{1}{12}$ sq. units

Answer is **E**.

10



a $A = -\int_0^2 (x^2 - 4)\mathrm{d}x$

$= -\left[\frac{x^3}{3} - 4x\right]_0^2$

$= -\left(\frac{8}{3} - 8\right)$

$= -\left(-\frac{16}{3}\right)$

$= 5\frac{1}{3}$ sq. units

b $A = \int_2^4 (x^2 - 4)\mathrm{d}x$

$= \left[\frac{x^3}{3} - 4x\right]_2^4$

$= \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)$

$= \frac{56}{3} - 8$

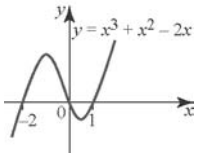
$= \frac{32}{3}$

$= 10\frac{2}{3}$ sq. units

c $A = 5\frac{1}{3} + 10\frac{2}{3}$

$= 16$ sq. units

11



a $A = \int_{-2}^0 (x^3 + x^2 - 2x)\mathrm{d}x$

$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_{-2}^0$

$= 0 - \left(4 - \frac{8}{3} - 4\right)$

$= \frac{8}{3}$

$= 2\frac{2}{3}$ sq. units

b $A = -\int_0^1 (x^3 + x^2 - 2x)\mathrm{d}x$

$= -\left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_0^1$

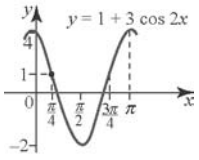
$= -\left[\frac{1}{4} + \frac{1}{3} - 1\right]$

$= \frac{5}{12}$ sq. units

c $A = 2\frac{2}{3} + \frac{5}{12}$

$= 3\frac{1}{12}$ sq. units

12



a $A = \int_0^{\frac{\pi}{4}} (1 + 3\cos 2x)\mathrm{d}x$

$= \left[x + \frac{3}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{\pi}{4} + \frac{3}{2}\sin\frac{\pi}{2}\right) - \left(0 + \frac{3}{2}\sin 0\right)$

$= \left(\frac{\pi}{4} + \frac{3}{2}\right)$ sq. units

b $A = \int_{\frac{3\pi}{4}}^{\pi} (1 + 3\cos 2x)\,dx$

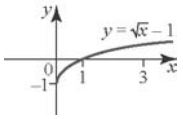
$= \left[x + \frac{3}{2}\sin 2x\right]_{\frac{3\pi}{4}}^{\pi}$

$= \left(\pi + \frac{3}{2}\sin 2\pi\right) - \left(\frac{3\pi}{4} + \frac{3}{2}\sin\frac{3\pi}{2}\right)$

$= \pi - \frac{3\pi}{4} + \frac{3}{2}$

$= \frac{\pi}{4} + \frac{3}{2}$

13



$A = \int_2^3 \left(\sqrt{x} - 1\right)dx$

$= \int_2^3 \left(x^{\frac{1}{2}} - 1\right)dx$

$= \left[\frac{2x^{\frac{3}{2}}}{3} - x\right]_2^3$

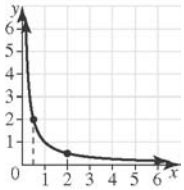
$= \left(\frac{2\left(\sqrt{3}\right)^3}{3} - 3\right) - \left(\frac{2\left(\sqrt{2}\right)^3}{3} - 2\right)$

$= \frac{2\times 3\sqrt{3}}{3} - 3 - \frac{4\sqrt{2}}{3} + 2$

$= \left(2\sqrt{3} - 1 - \frac{4\sqrt{2}}{3}\right)$ sq. units

$\simeq 0.578$ sq. units

14



$A = \int_{\frac{1}{2}}^{2} \frac{1}{x}dx$

$= \left[\log_e x\right]_{\frac{1}{2}}^{2}$

$= \log_e 2 - \log_e \frac{1}{2}$

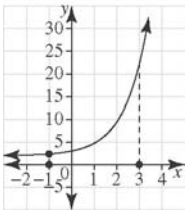
$= \log_e \left(\frac{2}{\frac{1}{2}}\right)$

$= \log_e 4$

$= \log_e 2^2$

$= 2\log_e 2$ sq. units

15



$A = \int_{-1}^{3} (e^x + 2)dx$

$= \left[e^x + 2x\right]_{-1}^{3}$

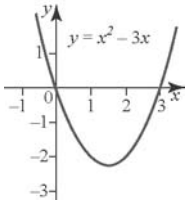
$= (e^3 + 6) - (e^{-1} - 2)$

$= (e^3 - e^{-1} + 8)$ sq. units.

Exercise 8C — Further areas

1 a $f(x) = x^2 - 3x$

i



ii $A = -\int_0^3 (x^2 - 3x)dx$

$= -\left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3$

$= -\left[9 - \frac{27}{2}\right]$

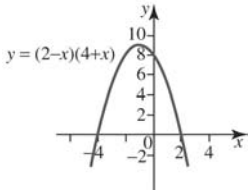
$= -\left(-\frac{9}{2}\right)$

$= 4\frac{1}{2}$ sq. units

b $g(x) = (2 - x)(4 + x)$

$= 8 - 2x - x^2$

i



ii $A = \int_{-4}^{2} (8 - 2x - x^2)dx$

$= \left[8x - x^2 - \frac{x^3}{3}\right]_{-4}^{2}$

$= \left(16 - 4 - \frac{8}{3}\right) - \left(-32 - 16 + \frac{64}{3}\right)$

$= 60 - \frac{72}{3}$

$= 60 - 24$

$= 36$ sq. units.

2 Sketch graph on graphics calculator to verify x-intercepts and terminals for each of the parts in this question

a $h(x) = (x + 3)(5 - x)$

x-intercepts at $x = -3$ and 5, positive area

$A = \int_{-3}^{5} (x + 3)(5 - x)dx$

$= \int_{-3}^{5} (-x^2 + 2x + 15)dx$

$$= \left[\frac{-x^3}{3} + x^2 + 15x\right]_{-3}^{5}$$

$$= \left(\frac{-125}{3} + 25 + 75\right) - (9 + 9 - 45)$$

$$= 85\frac{1}{3} \text{ sq. units}$$

b $h(x) = x^2 + 5x - 6$

$= (x + 6)(x - 1)$

x-intercepts at $x = -6$ and 1, negative area

$$A = -\int_{-6}^{1}(x^2 + 5x - 6)\,dx$$

$$= -\left[\frac{x^3}{3} + \frac{5x^2}{2} - 6x\right]_{-6}^{1}$$

$$= -\left[\left(\frac{1}{3} + \frac{5}{2} - 6\right) - \left(\frac{-216}{3} + 90 + 36\right)\right]$$

$$= -\left(\frac{-343}{6}\right)$$

$$= 57\frac{1}{6} \text{ sq. units}$$

c $g(x) = 8 - x^2$

$= \left(\sqrt{8} - x\right)\left(\sqrt{8} + x\right)$

x-intercepts at $x = \pm\sqrt{8}$, positive area

$$A = \int_{-\sqrt{8}}^{\sqrt{8}}(8 - x^2)\,dx$$

$$= \left[8x - \frac{x^3}{3}\right]_{-\sqrt{8}}^{\sqrt{8}}$$

$$= \left(8\sqrt{8} - \frac{8\sqrt{8}}{3}\right) - \left(-8\sqrt{8} + \frac{8\sqrt{8}}{3}\right)$$

$$= 16\sqrt{8} - \frac{16\sqrt{8}}{3}$$

$$= 32\sqrt{2} - \frac{32\sqrt{2}}{3}$$

$$= \frac{96\sqrt{2} - 32\sqrt{2}}{3}$$

$$= \frac{64\sqrt{2}}{3}$$

d $g(x) = x^3 - 4x^2$

$= x^2(x - 4)$

x-intercepts at $x = 0$ and 4, negative area

$$A = -\int_{0}^{4}(x^3 - 4x^2)\,dx$$

$$= -\left[\frac{x^4}{4} - \frac{4x^3}{3}\right]_{0}^{4}$$

$$= -\left[\left(64 - \frac{256}{3}\right) - 0\right]$$

$$= -\left(-21\frac{1}{3}\right)$$

$$= 21\frac{1}{3} \text{ sq. units}$$

e $f(x) = x(x - 2)(x - 3)$

$= x^3 - 5x^2 + 6x$

x-intercepts at $x = 0$, 2 and 3

positive area between $x = 0$ and 2

negative area between $x = 2$ and 3

$$A = \int_{0}^{2}(x^3 - 5x^2 + 6x)\,dx - \int_{2}^{3}(x^3 - 5x^2 + 6x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2\right]_{0}^{2} - \left[\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2\right]_{2}^{3}$$

$$= \left(4 - \frac{40}{3} + 12\right) - \left(\left(\frac{81}{4} - 45 + 27\right) - \left(4 - \frac{40}{3} + 12\right)\right)$$

$$= \frac{8}{3} - \left(\frac{9}{4} - \frac{8}{3}\right)$$

$$= \frac{16}{3} - \frac{9}{4}$$

$$= \frac{64 - 27}{12}$$

$$= \frac{37}{12}$$

$$= 3\frac{1}{12} \text{ sq. units}$$

f $f(x) = x^3 - 4x^2 - 4x + 16$

$= (x - 4)(x - 2)(x + 2)$

x-intercepts are at $x = -2$, 2 and 4

positive area between $x = -2$ and 2

negative area between $x = 2$ and 4

$$A = \int_{-2}^{2}(x^3 - 4x^2 - 4x + 16)\,dx - \int_{2}^{4}(x^3 - 4x^2 - 4x + 16)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{4x^3}{3} - 2x^2 + 16x\right]_{-2}^{2} - \left[\frac{x^4}{4} - \frac{4x^3}{3} - 2x^2 + 16x\right]$$

$$= \left(4 - \frac{32}{3} - 8 + 32\right) - \left(4 + \frac{32}{3} - 8 - 32\right)$$

$$-\left[\left(64 - \frac{256}{3} - 32 + 64\right) - \left(4 - \frac{32}{3} - 8 + 32\right)\right]$$

$$= \left(\frac{52}{3}\right) - \left(\frac{-76}{3}\right) - \left[\left(\frac{32}{3}\right) - \left(\frac{52}{3}\right)\right]$$

$$= \frac{148}{3}$$

$$= 49\frac{1}{3} \text{ sq. units}$$

g $g(x) = x^3 + 3x^2 - x - 3$

$= (x - 1)(x + 1)(x + 3)$

x-intercepts at $x = -3$, -1 and 1

positive area between $x = -3$ and -1

negative area between $x = -1$ and 1

$$A = \int_{-3}^{-1}(x^3 + 3x^2 - x - 3)\,dx - \int_{-1}^{1}(x^3 + 3x^2 - x - 3)\,dx$$

$$= \left[\frac{x^4}{4} + x^3 - \frac{x^2}{2} - 3x\right]_{-3}^{-1} - \left[\frac{x^4}{4} + x^3 - \frac{x^2}{2} - 3x\right]_{-1}^{1}$$

$$= \left(\frac{1}{4} - 1 - \frac{1}{2} + 3\right) - \left(\frac{81}{4} - 27 - \frac{9}{2} + 9\right)$$

$$-\left[\left(\frac{1}{4} + 1 - \frac{1}{2} - 3\right) - \left(\frac{1}{4} - 1 - \frac{1}{2} + 3\right)\right]$$

$$= \left(\frac{7}{4}\right) - \left(\frac{-9}{4}\right) - \left[\left(\frac{-9}{4}\right) - \left(\frac{7}{4}\right)\right]$$

$= 4 + 4$

$= 8$ sq. units

h $h(x) = (x - 1)(x + 2)(x + 5)$

$= x^3 + 6x^2 + 3x - 10$

x-intercepts at $x = -5$, -2 and 1

positive area between $x = -5$ and -2

negative area between $x = -2$ and 1

$$A = \int_{-5}^{-2}(x^3+6x^2+3x-10)\mathrm{d}x - \int_{-2}^{1}(x^3+6x^2+3x-10)\mathrm{d}x$$
$$= \left[\frac{x^4}{4}+2x^3+\frac{3x^2}{2}-10x\right]_{-5}^{-2} - \left[\frac{x^4}{4}+2x^3+\frac{3x^2}{2}-10x\right]_{-2}^{1}$$
$$= \left(4-16+6+20\right)-\left(\frac{625}{4}-250+\frac{75}{2}+50\right)$$
$$= -\left[\left(\frac{1}{4}+2+\frac{3}{2}-10\right)-\left(4-16+6+20\right)\right]$$
$$= (14)-\left(\frac{-25}{4}\right)-\left[\left(\frac{-25}{5}\right)-14\right]$$
$$= \frac{81}{4}-\left(\frac{-81}{4}\right)$$
$$= \frac{81}{2}$$
$$= 40\frac{1}{2} \text{ sq. units}$$

3 $y = x^2 - 6x + 8$
$= (x-4)(x-2)$
x-intercepts are at $x = 4$ and 2, negative area

$$A = -\int_2^4 (x^2-6x+8)\mathrm{d}x$$
$$= -\left[\frac{x^3}{3}-3x^2+8x\right]_2^4$$
$$= -\left[\left(\frac{64}{3}-48+32\right)-\left(\frac{8}{3}-12+16\right)\right]$$
$$= -\left(\frac{16}{3}+\frac{20}{3}\right)$$
$$= \frac{4}{3} \text{ sq. units}$$

Answer is **A**

4 $A = -\int_{-3}^{2} f(x)\mathrm{d}x + \int_2^4 f(x)\mathrm{d}x$
$= \int_2^4 f(x)\mathrm{d}x - \int_{-3}^{2} f(x)\mathrm{d}x$

Answer is **D**

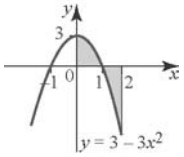
5 $y = x^2 - x - 6$
$= (x-3)(x+2)$
x-intercepts at $x = -2$ and 3
negative area between intercepts
Use a calculator to sketch the graph

$$A = -\int_2^3 (x^2-x-6)\mathrm{d}x + \int_3^4 (x^2-x-6)\mathrm{d}x$$
$$= -\left[\frac{x^3}{3}-\frac{x^2}{2}-6x\right]_2^3 + \left[\frac{x^3}{3}-\frac{x^2}{2}-6x\right]_3^4$$
$$= -\left[\left(9-\frac{9}{2}-18\right)-\left(\frac{8}{3}-2-12\right)\right]+\left(\frac{64}{3}-8-24\right)$$
$$-\left(9-\frac{9}{2}-18\right)$$
$$= -\left[\left(\frac{-27}{2}\right)-\left(\frac{-34}{3}\right)\right]+\left(\frac{-32}{3}\right)-\left(\frac{-27}{2}\right)$$
$$= \frac{13}{6}+\frac{17}{6}$$
$$= 5 \text{ sq. units}$$

Answer is **C**

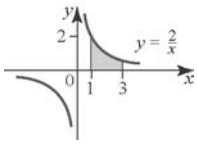
6 Use a graphics calculator with a suitable domain to sketch each of the following parts **a** to **j**.

a **i** $y = 3 - 3x^2$, $x = 0$, $x = 2$



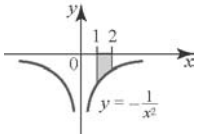
ii $A = \int_0^1 (3-3x^2)\mathrm{d}x - \int_1^2 (3-3x^2)\mathrm{d}x$
$= \left[3x-x^3\right]_0^1 - \left[3x-x^3\right]_1^2$
$= ((3-1)-0)-((6-8)-(3-1))$
$= 2-(-4)$
$= 6$ sq. units

b **i** $y = \frac{2}{x}$, $x = 1$, $x = 3$



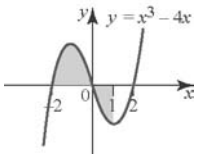
ii $A = \int_1^3 \frac{2}{x}\mathrm{d}x$
$= \left[2\log_e x\right]_1^3$
$= 2\log_e 3$ sq. units

c **i** $y = \frac{-1}{x^2}$, $x = 1$, $x = 2$



ii $A = -\int_1^2 -\frac{1}{x^2}\mathrm{d}x$
$= -\int_1^2 -x^{-2}\mathrm{d}x$
$= -\left[x^{-1}\right]_1^2$
$= -\left(\frac{1}{2}-\frac{1}{1}\right)$
$= -\left(-\frac{1}{2}\right)$
$= \frac{1}{2}$

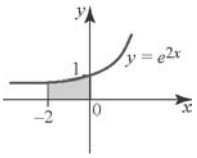
d **i** $y = x^3 - 4x$, $x = -2$, $x = 1$



ii $A = \int_{-2}^0 (x^3-4x)\mathrm{d}x - \int_0^1 (x^3-4x)\mathrm{d}x$
$= \left[\frac{x^4}{4}-2x^2\right]_{-2}^0 - \left[\frac{x^4}{4}-2x^2\right]_0^1$
$= \left(0-(4-8)\right)-\left(\left(\frac{1}{4}-2\right)-0\right)$
$= 4-\left(\frac{-7}{4}\right)$
$= \frac{23}{4}$

$= 5\frac{3}{4}$ sq. units

e i $y = e^{2x}, x = -2, x = 0$



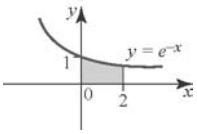
ii $A = \int_{-2}^{0} e^{2x}\,dx$

$= \left[\frac{1}{2}e^{2x}\right]_{-2}^{0}$

$= \frac{1}{2} - \frac{1}{2}e^{-4}$

$= \frac{1}{2}(1 - e^{-4})$ sq. units

f i $y = e^{-x}, x = 0, x = 2$



ii $A = \int_{0}^{2} e^{-x}\,dx$

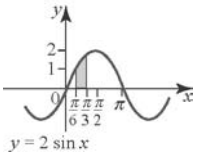
$= \left[-e^{-x}\right]_{0}^{2}$

$= -e^{-2} + e^{0}$

$= -e^{-2} + 1$

$= 1 - e^{-2}$ sq. units

g i $y = 2\sin x, x = \frac{\pi}{6}, x = \frac{\pi}{3}$



ii $A = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sin x\,dx$

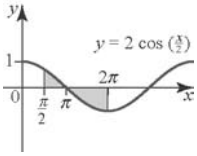
$= \left[-2\cos x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= -2\cos\frac{\pi}{3} - -2\cos\frac{\pi}{6}$

$= -1 + \sqrt{3}$

$= \sqrt{3} - 1$ sq. units

h i $y = \cos\frac{x}{2}, x = \frac{\pi}{2}, x = 2\pi$



ii $A = \int_{\frac{\pi}{2}}^{\pi} \cos\frac{x}{2}\,dx - \int_{\pi}^{2\pi} \cos\frac{x}{2}\,dx$

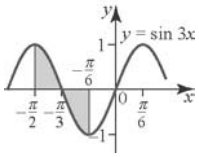
$= \left[2\sin\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi} - \left[2\sin\frac{x}{2}\right]_{\pi}^{2\pi}$

$= \left(2\sin\frac{\pi}{2} - 2\sin\frac{\pi}{4}\right) - \left(2\sin\pi - 2\sin\frac{\pi}{2}\right)$

$= (2 - \sqrt{2}) - (0 - 2)$

$= 4 - \sqrt{2}$ sq. units

i i $y = \sin 3x, x = -\frac{\pi}{2}, x = -\frac{\pi}{6}$



ii $A = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} \sin 3x\,dx - \int_{-\frac{\pi}{3}}^{-\frac{\pi}{6}} \sin 3x\,dx$

$= \left[-\frac{1}{3}\cos 3x\right]_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} - \left[-\frac{1}{3}\cos 3x\right]_{-\frac{\pi}{3}}^{-\frac{\pi}{6}}$

$= \left(-\frac{1}{3}\cos(-\pi) - \frac{-1}{3}\cos\left(\frac{-3\pi}{2}\right)\right) -$

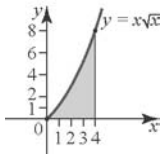
$\left(-\frac{1}{3}\cos\left(-\frac{\pi}{2}\right) - \frac{-1}{3}\cos(-\pi)\right)$

$= \left(\frac{1}{3} - 0\right) - \left(0 - \frac{1}{3}\right)$

$= \frac{2}{3}$ sq. units

j i $y = x\sqrt{x}, x \geq 0, x = 0, x = 4$

$= x^{\frac{3}{2}}$



ii $A = \int_{0}^{4} x^{\frac{3}{2}}\,dx$

$= \left[\frac{2x^{\frac{5}{2}}}{5}\right]_{0}^{4}$

$= \frac{2(2)^5}{5}$

$= \frac{64}{5}$

$= 12\frac{4}{5}$ sq. units

7 Use a graphics calculator with a suitable domain to sketch each of the following parts **a** to **g**

a i $y = x - 4x^{-1}, x \neq 0, x = 1, x = 3$

$y = x - \frac{4}{x}$

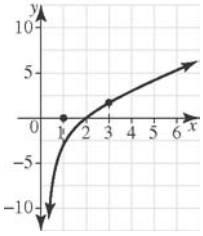
at $y = 0, x - \frac{4}{x} = 0$

$x^2 - 4 = 0$

$(x - 2)(x + 2) = 0$

$x = 2, -2$

x-intercept at $x = 2$ for this domain



ii $A = -\int_1^2 \left(x - \frac{4}{x}\right) dx + \int_2^3 \left(x - \frac{4}{x}\right) dx$

$= -\left[\frac{x^2}{2} - 4\log_e x\right]_1^2 + \left[\frac{x^2}{2} - 4\log_e x\right]_2^3$

$= -\left[(2 - 4\log_e 2) - \left(\frac{1}{2} - 0\right)\right] +$

$\left[\left(\frac{9}{2} - 4\log_e 3\right) - (2 - 4\log_e 2)\right]$

$= -\left[2 - 4\log_e 2 - \frac{1}{2}\right] + \left[\frac{9}{2} - 4\log_e 3 - 2 + 4\log_e 2\right]$

$= -\frac{3}{2} + 4\log_e 2 + \frac{5}{2} - 4\log_e 3 + 4\log_e 2$

$= 1 + \log_e \frac{2^4 \times 2^4}{3^4}$

$= 1 + \log_e \left(\frac{4}{3}\right)^4$

$= 1 + 4\log_e \left(\frac{4}{3}\right)$

$= 1 - 4\log_e \left(\frac{3}{4}\right)$

$\approx 2{\cdot}15$

b i $y = \sin x - \cos x \quad 0 \le x \le \pi$

$x = 0, x = \pi$

at $y = 0$, $\sin x - \cos x = 0$

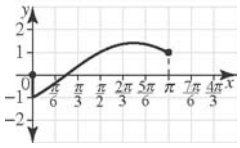
$\sin x = \cos x$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$x = \frac{\pi}{4}$

x-intercept at $x = \frac{\pi}{4}$



ii $A = -\int_0^{\frac{\pi}{4}} (\sin x - \cos x)\, dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x) dx$

$= -[-\cos x - \sin x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\pi}$

$= -\left[\left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) - (-\cos 0 - \sin 0)\right] +$

$\left[(-\cos\pi - \sin\pi) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)\right]$

$= -\left[\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) - (-1)\right] + \left[1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right]$

$= -(-\sqrt{2} + 1) + (1 + \sqrt{2})$

$= \sqrt{2} - 1 + 1\sqrt{2}$

$= 2\sqrt{2}$ sq. units

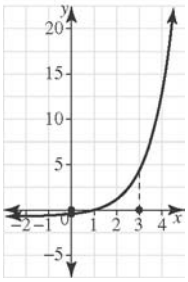
c i $y = e^x - e, x = 0, x = 3$

at $y = 0$, $e^x - e = 0$

$e^x = e^1$

$x = 1$

x-intercept at $x = 1$



ii $A = -\int_0^1 \left(e^x - e\right) dx + \int_1^3 \left(e^x - e\right) dx$

$= -\left[e^x - ex\right]_0^1 + \left[e^x - ex\right]_1^3$

$= -((e - e) - e^0) + (e^3 - 3e) - (e^1 - e)$

$= 1 + e^3 - 3e$ sq. units

$\simeq 12{\cdot}93$

d i $y = x - \frac{1}{x^2}, x \neq 0, x = \frac{1}{2}, x = 2$

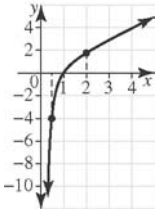
at $y = 0$, $x - \frac{1}{x^2} = 0$

$x^3 - 1 = 0$

$x^3 = 1$

$x = 1$

x-intercept at $x = 1$



ii $A = -\int_{\frac{1}{2}}^1 \left(x - \frac{1}{x^2}\right) dx + \int_1^2 \left(x - \frac{1}{x^2}\right) dx$

$= -\left[\frac{x^2}{2} + \frac{1}{x}\right]_{\frac{1}{2}}^1 + \left[\frac{x^2}{2} + \frac{1}{x}\right]_1^2$

$= -\left(\left(\frac{1}{2} + 1\right) - \left(\frac{1}{8} + 2\right)\right) + \left(\left(2 + \frac{1}{2}\right) - \left(\frac{1}{2} + 1\right)\right)$

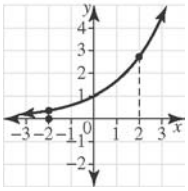
$= \frac{5}{8} + 1$

$= 1\frac{5}{8}$ sq. units

e i $y = e^{\frac{x}{2}}, x = -2, x = 2$

at $y = 0$, $e^{\frac{x}{2}} = 0$

no x-intercepts



ii $A = \int_{-2}^2 e^{\frac{x}{2}}\, dx$

$= \left[2e^{\frac{x}{2}}\right]_{-2}^2$

$= 2e^1 - 2e^{-1}$

$\simeq 4{\cdot}7$ sq. units

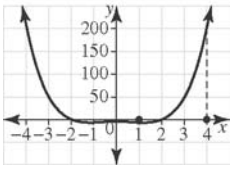
f **i** $y = x^4 - 3x^2 - 4$, $x = 1$, $x = 4$

at $y = 0$, $x^4 - 3x^2 - 4 = 0$

$(x^2 - 4)(x^2 + 1) = 0$

$(x - 2)(x + 2)(x^2 + 1) = 0$

x-intercepts at $x = -2$, $x = 2$



ii $A = -\int_1^2 \left(x^4 - 3x^2 - 4\right) dx + \int_2^4 \left(x^4 - 3x^2 - 4\right) dx$

$= -\left[\frac{x^5}{5} - x^3 - 4x\right]_1^2 + \left[\frac{x^5}{5} - x^3 - 4x\right]_2^4$

$= -\left[\left(\frac{32}{5} - 8 - 8\right) - \left(\frac{1}{5} - 1 - 4\right)\right] +$

$\left[\left(\frac{1024}{5} - 64 - 16\right) - \left(\frac{32}{5} - 8 - 8\right)\right]$

$= \frac{24}{5} + \frac{672}{5}$

$= \frac{696}{5}$

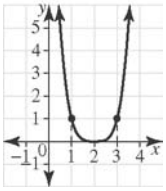
$= 139{\cdot}2$ sq. units

g **i** $y = (x - 2)^4$, $x = 1$, $x = 3$

at $y = 0$, $(x - 2)^4 = 0$

$x = 2$

x-intercept at $x = 2$.



ii $A = \int_1^2 (x-2)^4 dx + \int_2^3 (x-2)^4 dx$

$= \left[\frac{(x-2)^5}{5}\right]_1^2 + \left[\frac{(x-2)^5}{5}\right]_2^3$

$= \left(0 - \frac{-1}{5}\right) + \left(\frac{1}{5} - 0\right)$

$= \frac{2}{5}$ sq. units

8 $y = e^{3x}$

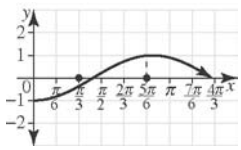
$A = \int_1^2 e^{3x}\, dx$

$= \left[\frac{1}{3}e^{3x}\right]_1^2$

$= \frac{1}{3}e^6 - \frac{1}{3}e^3$

$= \frac{1}{3}(e^6 - e^3)$ sq. units

9 $y = -\cos x$, $x = \frac{\pi}{3}$, $x = \frac{5\pi}{6}$



$A = -\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} -\cos x\, dx + \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} -\cos x\, dx$

$= -\left[-\sin x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \left[-\sin x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$

$= -\left((-1) - \left(-\frac{\sqrt{3}}{2}\right)\right) + \left(-\frac{1}{2} - -1\right)$

$= 1 - \frac{\sqrt{3}}{2} - \frac{1}{2} + 1$

$= \frac{3 - \sqrt{3}}{2}$ sq. units

10 $y = (x - 1)^3$

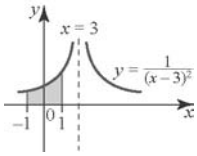
$A = -\int_0^1 (x-1)^3 dx$

$= -\left[\frac{(x-1)^4}{4}\right]_0^1$

$= -\left[0 - \frac{1}{4}\right]$

$= \frac{1}{4}$ sq. units

11 **a**



b $A = \int_{-1}^1 \frac{1}{(x-3)^2} dx$

$= \int_{-1}^1 (x-3)^{-2} dx$

$= \left[\frac{(x-3)^{-1}}{-1}\right]_{-1}^1$

$= \left[-\frac{1}{(x-3)}\right]_{-1}^1$

$= \left[\frac{1}{3-x}\right]_{-1}^1$

$= \left(\frac{1}{2} - \frac{1}{4}\right)$

$= \frac{1}{4}$ sq. units

12 **a** $f(x) = (x + 2)^{-3}$

$= \frac{1}{(x+2)^3}$

asymptotes at $x = -2$ and $y = 0$.

b $A = \int_{-1}^1 (x+2)^{-3} dx$

$= \left[\frac{(x+2)^{-2}}{-2}\right]_{-1}^1$

$= \left[\frac{-1}{2(x+2)^2}\right]_{-1}^1$

$= \left(-\frac{1}{18} - \frac{-1}{2}\right)$

$= \frac{4}{9}$ sq. units

13 $y = 3 - e^{2x}$

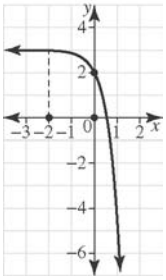
at $y = 0$, $3 - e^{2x} = 0$

$e^{2x} = 3$

$\log_e 3 = 2x$

$x = \frac{1}{2}\log_e 3$

$x \simeq 0{\cdot}55$



$A = \int_{-2}^{0}\left(3 - e^{2x}\right)dx$

$= \left[3x - \frac{1}{2}e^{2x}\right]_{-2}^{0}$

$= \left(-\frac{1}{2}\right) - \left(-6 - \frac{1}{2}e^{-4}\right)$

$= \frac{11}{2} + \frac{e^{-4}}{2}$

$= \frac{11 + e^{-4}}{2}$ sq. units.

14 $y = 4 - \sin 2x$

at $y = 0$, $4 - \sin 2x = 0$

$4 = \sin 2x$

no x-intercepts.

$A = \int_{-\frac{\pi}{2}}^{\pi}(4 - \sin 2x)\,dx$

$= \left[4x + \frac{1}{2}\cos 2x\right]_{-\frac{\pi}{2}}^{\pi}$

$= \left(4\pi + \frac{1}{2}\right) - \left(-2\pi - \frac{1}{2}\right)$

$= 6\pi + 1$ sq. units.

15 a $y = x\log_e x \quad (x > 0)$

$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \log_e x$

$= 1 + \log_e x$

b $\int(1 + \log_e x)dx = x\log_e x$

$\int 1\,dx + \int \log_e x\,dx = x\log_e x$

$\int \log_e x\,dx = x\log_e x - \int 1\,dx$

$= x\log_e x - x$

c $\int_1^4 \log_e x\,dx = \left[x\log_e x - x\right]_1^4$

$= (4\log_e 4 - 4) - (\log_e 1 - 1)$

$= 4\log_e 4 - 3$ sq. units

16 a $y = \log_e(x^2 + 2)$

$\frac{dy}{dx} = \frac{2x}{x^2 + 2}$

b $\int \frac{2x}{x^2 + 2}dx = \log_e(x^2 + 2)$

$2\int \frac{x}{x^2 + 2}dx = \log_e(x^2 + 2)$

$\int \frac{x}{x^2 + 2}dx = \frac{1}{2}\log_e(x^2 + 2)$

c x-intercept at $x = 0$

$A = -\int_{-1}^{0}\frac{x}{x^2 + 2}dx + \int_0^1 \frac{x}{x^2 + 2}dx$

$= -\left[\frac{1}{2}\log_e\left(x^2 + 2\right)\right]_{-1}^{0} + \left[\frac{1}{2}\log_e\left(x^2 + 2\right)\right]_0^1$

$= -\left(\frac{1}{2}\log_e 2 - \frac{1}{2}\log_e 3\right) + \left(\frac{1}{2}\log_e 3\right) - \left(\frac{1}{2}\log_e 2\right)$

$= \frac{1}{2}\log_e 3 - \frac{1}{2}\log_e 2 + \frac{1}{2}\log_e 3 - \frac{1}{2}\log_e 2$

$= \log_e\left(\frac{3^{\frac{1}{2}} \times 3^{\frac{1}{2}}}{2^{\frac{1}{2}} \times 2^{\frac{1}{2}}}\right)$

$= \log_e\left(\frac{3}{2}\right)$

$\simeq 0{\cdot}4055$ sq. units.

17 a $y = x^2$, $x = 0$, $x = 2$.

$A = \int_0^2 x^2\,dx$

$= \left[\frac{x^3}{3}\right]_0^2$

$= \frac{8}{3}$ sq. units.

b Area of rectangle $= 2 \times 4 = 8$ sq. units

$\Rightarrow$ Area of shaded region $= 8 - \frac{8}{3}$ sq. units

$= \frac{16}{3}$

$= 5\frac{1}{3}$ sq. units

18 $A_1 = \int_0^2 e^{2x}dx$

$= \left[\frac{1}{2}e^{2x}\right]_0^2$

$= \frac{1}{2}e^4 - \frac{1}{2}$ sq. units.

Area of rectangle $= 2 \times e^4 = 2e^4$ sq. units

Shaded area $= 2e^4 - \left(\frac{1}{2}e^4 - \frac{1}{2}\right)$

$= \frac{3e^4}{2} + \frac{1}{2}$

$= \frac{3e^4 + 1}{2}$

19 $A = \int_0^{\frac{x}{2}} 2\sin x\,dx$

$= \left[-2\cos x\right]_0^{\frac{\pi}{2}}$

$= 0 + 2$

$= 2$

Area of rectangle $= \frac{\pi}{2} \times 2 = \pi$

Shaded area $= \pi - 2$

Total shaded area $= (\pi - 2) \times 2$

$= 2\pi - 4$ sq. units.

Exercise 8D — Areas between two curves

1 and **2 a** $f(x) = x$
$g(x) = x^2$
$f(x) - g(x) = x - x^2$

$$\text{Area} = \int_0^1 \left(x - x^2\right) \mathrm{d}x$$
$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$
$$= \frac{1}{2} - \frac{1}{3}$$
$$= \frac{1}{6} \text{ sq. units}$$

b $f(x) = 2x$
$g(x) = x + 1$
$f(x) - g(x) = 2x - x - 1$
$= x - 1$

$$\text{Area} = \int_1^2 (x - 1)\,\mathrm{d}x$$
$$= \left[\frac{x^2}{2} - x\right]_1^2$$
$$= (2 - 2) - \left(\frac{1}{2} - 1\right)$$
$$= \frac{1}{2} \text{ sq. units}$$

c $f(x) = 4 - x^2$
$g(x) = 3x$
$f(x) - g(x) = 4 - x^2 - 3x$

$$\text{Area} = \int_0^1 \left(4 - x^2 - 3x\right) \mathrm{d}x$$
$$= \left[4x - \frac{x^3}{3} - \frac{3x^2}{2}\right]_0^1$$
$$= 4 - \frac{1}{3} - \frac{3}{2}$$
$$= 2\frac{1}{6} \text{ sq. units}$$

d $f(x) = 8 - x^2$
$g(x) = x^2$
$f(x) - g(x) = 8 - x^2 - x^2$
$= 8 - 2x^2$

$$\text{Area} = \int_{-2}^2 \left(8 - 2x^2\right) \mathrm{d}x$$
$$= \left[8x - \frac{2x^3}{3}\right]_{-2}^2$$
$$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$$
$$= 32 - \frac{32}{3}$$
$$= \frac{64}{3}$$
$$= 21\frac{1}{3} \text{ sq. units}$$

e $f(x) = 3x$
$g(x) = x^3$
$f(x) - g(x) = 3x - x^3$

$$\text{Area} = \int_0^{\sqrt{3}} \left(3x - x^3\right) \mathrm{d}x$$
$$= \left[\frac{3x^2}{2} - \frac{x^4}{4}\right]_0^{\sqrt{3}}$$
$$= \frac{9}{2} - \frac{9}{4}$$
$$= \frac{9}{4} = 2\frac{1}{4} \text{ sq. units}$$

f $f(x) = 9 - x^2$
$g(x) = e^x$
$f(x) - g(x) = 9 - x^2 - e^x$

$$\text{Area} = \int_{-1}^1 \left(9 - x^2 - e^x\right) \mathrm{d}x$$
$$= \left[9x - \frac{x^3}{3} - e^x\right]_{-1}^1$$
$$= \left(9 - \frac{1}{3} - e\right) - \left(-9 + \frac{1}{3} - e^{-1}\right)$$
$$= 18 - \frac{2}{3} - e + e^{-1}$$
$$= 17\frac{1}{3} - e + e^{-1}$$
$$\simeq 14{\cdot}98 \text{ sq. units}$$

g $f(x) = x$
$g(x) = -e^x$
$f(x) - g(x) = x - {}^-e^x$
$= x + e^x$

$$\text{Area} = \int_1^2 \left(x + e^x\right) \mathrm{d}x$$
$$= \left[\frac{x^2}{2} - e^x\right]_1^2$$
$$= (2 + e^2) - \left(\frac{1}{2} + e\right)$$
$$= 1\frac{1}{2} + e^2 - e$$
$$\simeq 6{\cdot}17 \text{ sq. units}$$

h $f(x) = -4$
$g(x) = x^2 - 5$
$f(x) - g(x) = -4 - x^2 + 5$
$= 1 - x^2$

$$\text{Area} = \int_{-1}^1 \left(1 - x^2\right) \mathrm{d}x$$
$$= \left[x - \frac{x^3}{3}\right]_{-1}^1$$
$$= \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)$$
$$= 2 - \frac{2}{3}$$
$$= \frac{4}{3} \text{ sq. units}$$

3 $A = \int_1^5 \left(g(x) - f(x)\right) \mathrm{d}x$ is D
$= \int_1^5 g(x)\,\mathrm{d}x - \int_1^5 f(x)\,\mathrm{d}x$ is A
$= \int_1^5 g(x)\,\mathrm{d}x + \int_5^1 f(x)\,\mathrm{d}x$ is B
$= \int_5^1 \left(f(x) - g(x)\right) \mathrm{d}x$ is E

Answer is **C**

4 $A = \int_{-3}^{-1}\big(f(x)-g(x)\big)\mathrm{d}x$

Answer is **D**

5 $A = \int_{0}^{3}\big(f(x)-g(x)\big)\mathrm{d}x + \int_{3}^{4}\big(g(x)-f(x)\big)\mathrm{d}x$

Answer is **E**

6 a i Intersection points when

$4x = x^2$

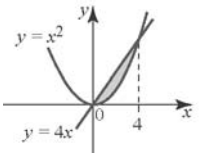
$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0$ and $x = 4$

ii Use calculator to check sketches.

$y = x^2$ and $y = 4x$



iii $A = \int_0^4 \left(4x - x^2\right)\mathrm{d}x$

$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$

$= 32 - \frac{64}{3}$

$= \frac{32}{3}$

$= 10\frac{2}{3}$ sq. units.

b i Intersection points when

$2x = 3 - x^2$

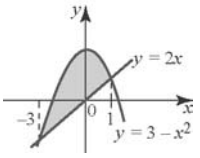
$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3$ or 1

ii Use calculator to check sketches.

$y = 2x$ and $y = 3 - x^2$



iii $A = \int_{-3}^{1}\left(3 - x^2 - 2x\right)\mathrm{d}x$

$= \left[3x - \frac{x^3}{3} - x^2\right]_{-3}^{1}$

$= \left(3 - \frac{1}{3} - 1\right) - (-9 + 9 - 9)$

$= 1\frac{2}{3} + 9$

$= 10\frac{2}{3}$ sq. units

c i Intersection points when

$x^2 - 1 = 1 - x^2$

$2x^2 - 2 = 0$

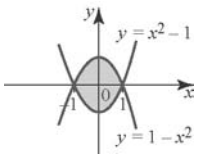
$2(x^2 - 1) = 0$

$2(x-1)(x+1) = 0$

$x = 1$ or -1

ii Use calculator to check sketches.

$y = x^2 - 1$ and $y = 1 - x^2$



iii $A = \int_{-1}^{1}(1 - x^2 - x^2 + 1)\mathrm{d}x$

$= \int_{-1}^{1}(2 - 2x^2)\mathrm{d}x$

$= \left[2x - \frac{2x^3}{3}\right]_{-1}^{1}$

$= \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right)$

$= 2\frac{2}{3}$ sq. units

d i Intersection points when

$x^2 - 4 = 4 - x^2$

$2x^2 - 8 = 0$

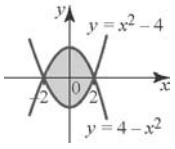
$2(x^2 - 4) = 0$

$2(x-2)(x+2) = 0$

$x = 2$ or -2

ii Use calculator to check sketches.

$y = x^2 - 4$ and $y = 4 - x^2$



iii $A = \int_{-2}^{2}\left(4 - x^2 - x^2 + 4\right)\mathrm{d}x$

$= \int_{-2}^{2}\left(8 - 2x^2\right)\mathrm{d}x$

$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$

$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$

$= 21\frac{1}{3}$ sq. units

e i Intersection points when

$(x+1)^2 = 1 - x^2$

$x^2 + 2x + 1 = 1 - x^2$

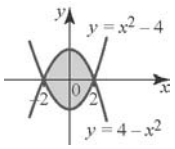
$2x^2 + 2x = 0$

$2x(x+1) = 0$

$x = 0$ and -1

ii Use calculator to check sketches.

$y = (x+1)^2$ and $y = 1 - x^2$



iii $A = \int_{-1}^{0} 1 - x^2 - (x+1)^2\,\mathrm{d}x$

$= \int_{-1}^{0}\left(1 - x^2 - x^2 - 2x - 1\right)\mathrm{d}x$

$= \int_{-1}^{0} -2x^2 - 2x\ \mathrm{d}x$

$$= \left[-\frac{2x^3}{3} - x^2\right]_{-1}^{0}$$

$$= 0 - \left(\frac{2}{3} - 1\right)$$

$$= \frac{1}{3} \text{ sq. units.}$$

f i Intersection points when

$$\sqrt{x} = x^2$$

$$x = x^4$$

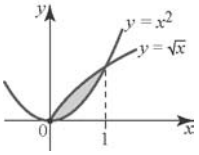
$$x^4 - x = 0$$

$$x(x^3 - 1) = 0$$

$$x = 0 \; x = 1$$

ii Use calculator to check sketch.

$y = \sqrt{x}$ and $y = x^2$



iii $A = \int_0^1 \sqrt{x} - x^2 \, dx$

$$= \int_0^1 \left(x^{\frac{1}{2}} - x^2\right) dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) - 0$$

$$= \frac{1}{3} \text{ sq. units}$$

7 a i $y = x^3$ and $y = x$

Intersection points when

$$x^3 = x$$

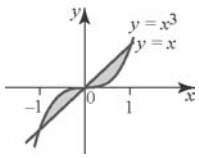
$$x^3 - x = 0$$

$$x\left(x^2 - 1\right) = 0$$

$$x(x - 1)(x + 1) = 0$$

$$x = -1, 0, 1$$

ii



iii $A = \int_{-1}^{0} \left(x^3 - x\right) dx + \int_0^1 \left(x - x^3\right) dx$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \left(0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right) + \left(\left(\frac{1}{2} - \frac{1}{4}\right) - 0\right)$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2} \text{ sq. units}$$

b i $y = 3x^2$ and $y = x^3 + 2x$

Intersection points when

$$3x^2 = x^3 + 2x$$

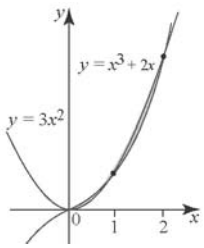
$$x^3 - 3x^2 + 2x = 0$$

$$x\left(x^2 - 3x + 2\right) = 0$$

$$x(x - 2)(x - 1) = 0$$

$$x = 0, 1, 2$$

ii



iii $A = \int_0^1 \left(x^3 + 2x - 3x^2\right) dx + \int_1^2 \left(3x^2 - x^3 - 2x\right) dx$

$$= \left[\frac{x^4}{4} + x^2 - x^3\right]_0^1 + \left[x^3 - \frac{x^4}{4} - x^2\right]_1^2$$

$$= \left(\frac{1}{4} + 1 - 1\right) + (8 - 4 - 4) - \left(1 - \frac{1}{4} - 1\right)$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2} \text{ sq. units}$$

8 a $y = x^3$, $y = x^2$, $-1 \le x \le 1$

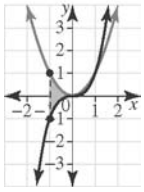
Intersection when

$$x^3 = x^2$$

$$x^3 - x^2 = 0$$

$$x^2(x - 1) = 0$$

$$x = 0, -1$$



$$A = \int_{-1}^{1} \left(x^2 - x^3\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{-1}^{1}$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{-1}{3} - \frac{1}{4}\right)$$

$$= \frac{2}{3} \text{ sq. units.}$$

b $y = \sin x$, $y = \cos x$ $0 \le x \le \pi$

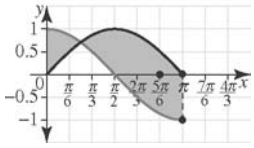
Intersection when

$$\sin x = \cos x$$

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

$$x = \frac{\pi}{4}$$



$$A=\int_0^{\frac{\pi}{4}}(\cos x-\sin x)\,dx+\int_{\frac{\pi}{4}}^{\pi}(\sin x-\cos x)\,dx$$

$$=[\sin x+\cos x]_0^{\frac{\pi}{4}}+[-\cos x-\sin x]_{\frac{\pi}{4}}^{\pi}$$

$$=\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)-(\sin 0+\cos 0)+$$

$$\left((-\cos\pi-\sin\pi)-\left(-\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\right)\right)$$

$$=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-0-1+1-0+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}$$

$$=2\sqrt{2}\text{ sq. units}$$

c $y=(x-1)^2,\ y=(x+1)^2,\ -1\le x\le 1$

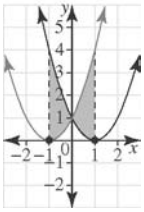
Intersection when

$$(x-1)^2=(x+1)^2$$

$$x^2-2x+1-x^2-2x-1=0$$

$$-4x=0$$

$$x=0$$



$$A=\int_1^0(x-1)^2-(x+1)^2\,dx+\int_0^1(x+1)^2-(x-1)^2\,dx$$

$$=\int_{-1}^0-4x\,dx+\int_0^{-1}4x\,dx$$

$$=\left[-2x^2\right]_{-1}^0+\left[2x^2\right]_0^1$$

$$=0-^-2+2$$

$$=4\text{ sq. units}$$

d $y=x^3-5x,\ y=6-2x^2,\ 0\le x\le 3$

Intersection when

$$x^3-5x=6-2x^2$$

$$x^3+2x^2-5x-6=0$$

$$(x-2)(x+1)(x+3)=0$$

$$x=-3,-1,2$$

$$A=\int_0^2\left(6-2x^2-x^3+5x\right)dx+\int_2^3\left(x^3-5x-6+2x^2\right)dx$$

$$=\left[6x-\frac{2x^3}{3}-\frac{x^4}{4}+\frac{5x^2}{2}\right]_0^2+\left[\frac{x^4}{4}-\frac{5x^2}{2}-6x+\frac{2x^3}{3}\right]_2^3$$

$$=\left(12-\frac{16}{3}-4+10\right)+\left(\frac{81}{4}-\frac{45}{2}-18+18\right)-$$

$$\left(4-10-12+\frac{16}{3}\right)$$

$$=18-\frac{16}{3}+\frac{81}{4}-\frac{45}{2}+18-\frac{16}{3}$$

$$=\frac{277}{12}$$

$$=23\frac{1}{12}\text{ sq. units}$$

e $y=\frac{1}{x},\ y=4x,\ \frac{1}{4}\le x\le 1$

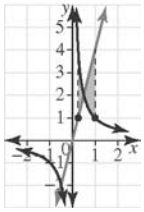
Intersection when

$$\frac{1}{x}=4x$$

$$1=4x^2$$

$$x^2=\frac{1}{4}$$

$$x=\pm\frac{1}{2}$$



$$A=\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\frac{1}{x}-4x\right)dx+\int_{\frac{1}{2}}^{1}\left(4x-\frac{1}{x}\right)dx$$

$$=\left[\log_e x-2x^2\right]_{\frac{1}{4}}^{\frac{1}{2}}+\left[2x^2-\log_e x\right]_{\frac{1}{2}}^{1}$$

$$=\left(\log_e\frac{1}{2}-\frac{1}{2}\right)-\left(\log_e\frac{1}{4}-\frac{1}{8}\right)+$$

$$\left(2-\log_e 1\right)-\left(\frac{1}{2}-\log_e\frac{1}{2}\right)$$

$$=\log_e\frac{1}{2}-\log_e\frac{1}{4}+\log_e\frac{1}{2}-\frac{1}{2}+\frac{1}{8}+2-\frac{1}{2}$$

$$=\log_e\left(\frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{4}}\right)+1\frac{1}{8}$$

$$=\log_e 1+1\frac{1}{8}$$

$$=1\frac{1}{8}\text{ sq. units}$$

f $y=e^x,\ y=e^{-x},\ 0\le x\le 1$

Intersection point

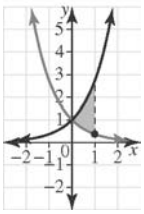
$$e^x=e^{-x}$$

$$\frac{e^x}{e^{-x}}=1$$

$$e^{2x}=1$$

$$2x=0$$

$$x=0$$



$$A=\int_0^1\left(e^x-e^{-x}\right)dx$$

$$=\left[e^x+e^{-x}\right]_0^1$$

$= \left(e^1 + e^{-1}\right) - \left(e^0 + e^0\right)$

$= e + e^{-1} - 2$.

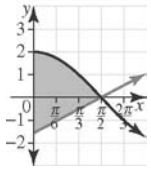
≈ 1.09 sq. units

g $y = 2\cos x,\ y = x - \frac{\pi}{2},\ 0 \le x \le \frac{\pi}{2}$

Intersection point

$2\cos x = x - \frac{\pi}{2}$

Use calculator to find intersection point, graph both curves and find intersection. Answer $= \left(\frac{\pi}{2}, 0\right)$



$A = \int_0^{\frac{\pi}{2}} \left(2\cos x - \left(x - \frac{\pi}{2}\right)\right) dx$

$= \left[2\sin x - \frac{x^2}{2} + \frac{\pi x}{2}\right]_0^{\frac{\pi}{2}}$

$= \left(2\sin\frac{\pi}{2} - \frac{\pi^2}{8} + \frac{\pi^2}{4}\right) - (2\sin 0 - 0 + 0)$

$= 2 - \frac{\pi^2}{8} + \frac{\pi^2}{4}$

$= 2 + \frac{\pi^2}{8}$ sq. units

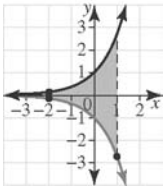
≈ 3.23 sq. units

h $y = e^x,\ y = -e^x,\ -2 \le x \le 1$

Intersection point

$e^x = -e^x$

no intersection



$A = \int_{-2}^{1} e^x + e^x dx = \int_{-2}^{1} 2e^x dx$

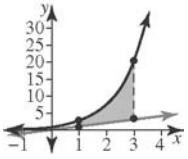
$= \left[2e^x\right]_{-2}^{1}$

$= 2e - 2e^{-2}$

$\simeq$ 5.17 sq. units

9 $y = e^x,\ y = x,\ x = 1,\ x = 3$

Use calculator to show no intersection between $x = 1$ and 3.



$A = \int_1^3 (e^x - x) dx$

$= \left[e^x - \frac{x^2}{2}\right]_1^3$

$= \left(e^3 - \frac{9}{2}\right) - \left(e^1 - \frac{1}{2}\right)$

$= e^3 - e^1 - 4$

$\simeq$ 13.37 sq. units

10 $y = x^2,\ y = \frac{x}{2} + 3,\ x = 1,\ x = 3$

Intersection when

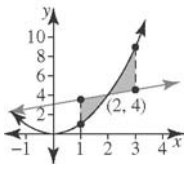
$x^2 = \frac{x}{2} + 3$

$x^2 - \frac{x}{2} - 3 = 0$

$2x^2 - x - 6 = 0$

$(2x + 3)(x - 2) = 0$

$x = \frac{-3}{2}$ or $x = 2$



$A = \int_1^2 \left(\frac{x}{2} + 3 - x^2\right) dx + \int_2^3 \left(x^2 - \frac{x}{2} - 3\right) dx$

$= \left[\frac{x^2}{4} + 3x - \frac{x^3}{3}\right]_1^2 + \left[\frac{x^3}{3} - \frac{x^2}{4} - 3x\right]_2^3$

$= \left(1 + 6 - \frac{8}{3}\right) - \left(\frac{1}{4} + 3 - \frac{1}{3}\right) + \left(9 - \frac{9}{4} - 9\right) - \left(\frac{8}{3} - 1 - 6\right)$

$= \frac{17}{12} + \frac{25}{12}$

$= \frac{42}{12} = \frac{7}{2}$

$= 3\frac{1}{2}$ sq. units

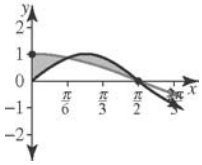
11 $y = \sin 2x,\ y = \cos x,\ x = 0$ to $x = \frac{\pi}{2}$

Intersection point when

$\sin 2x = \cos x$

Use graphics calculator to find the intersection between the two curves.

Answer is $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$



Area $= \int_0^{\frac{\pi}{6}} (\cos x - \sin 2x) dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin 2x - \cos x) dx$

$= \left[\sin x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}} + \left[\frac{-1}{2}\cos 2x - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \left(\sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3}\right) - \left(\sin 0 + \frac{1}{2}\cos 0\right) +$

$\left(\frac{-1}{2}\cos\pi - \sin\frac{\pi}{2}\right) - \left(\frac{-1}{2}\cos\frac{\pi}{3} - \sin\frac{\pi}{6}\right)$

$= \frac{1}{2} + \frac{1}{4} - 0 - \frac{1}{2} + \frac{1}{2} - 1 + \frac{1}{4} + \frac{1}{2}$

$= \frac{1}{2}$ sq. units

12 $y = \sqrt{3} - \sin 2x,\ y = \sin 2x,\ x = 0 \text{ to } \dfrac{\pi}{4}$

Intersection point when

$\sqrt{3} - \sin 2x = \sin 2x$

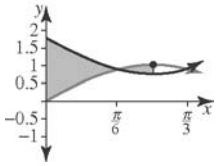
$2 \sin 2x = \sqrt{3}$

$\sin 2x = \dfrac{\sqrt{3}}{2}$

basic angle $= \dfrac{\pi}{3}$

$2x = \dfrac{\pi}{3}$

$x = \dfrac{\pi}{6}$



$$\text{Area} = \int_0^{\frac{\pi}{6}} \left(\sqrt{3} - 2\sin 2x\right) dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(2\sin 2x - \sqrt{3}\right) dx$$

$$= \left[\sqrt{3}x + \cos 2x\right]_0^{\frac{\pi}{6}} + \left[-\cos 2x - \sqrt{3}x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \left(\frac{\sqrt{3}\pi}{6} + \cos\frac{\pi}{3}\right) - (\cos 0) + \left(-\cos\frac{\pi}{2} - \frac{\sqrt{3}\pi}{4}\right) - \left(-\cos\frac{\pi}{3} - \frac{\sqrt{3}\pi}{6}\right)$$

$$= \frac{\sqrt{3}\pi}{6} + \frac{1}{2} - 1 - 0 - \frac{\sqrt{3}\pi}{4} + \frac{1}{2} + \frac{\sqrt{3}\pi}{6}$$

$$= \frac{\sqrt{3}\pi}{3} - \frac{\sqrt{3}\pi}{4}$$

$$= \frac{\sqrt{3}\pi}{12}$$

$\simeq 0.45$ sq. units

13 $y = e^x,\ y = 3 - 2e^{-x}$

Intersection when

$e^x = 3 - 2e^{-x}$

$e^x + \dfrac{2}{e^x} - 3 = 0$

$e^{2x} + 2 - 3e^x = 0$

Let $a = e^x$

$a^2 - 3a + 2 = 0$

$(a - 2)(a - 1) = 0$

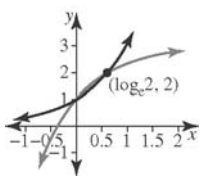
$a = 1$ or 2

at $a = 1$, $e^x = 1$

$x = 0$

at $a = 2$, $e^x = 2$

$x = \log_e 2$



$$\text{Area} = \int_0^{\log_e 2} \left(3 - 2e^{-x} - e^x\right) dx$$

$$= \left[3x + 2e^{-x} - e^{-x}\right]_0^{\log_e 2}$$

$$= (3\log_e 2 + 2e^{-\log_e 2} - e^{\log_e 2}) - (0 + 2e^0 - e^0)$$

$$= 3\log_e 2 + 2 \times \frac{1}{2} - 2 - 2 + 1$$

$$= 3\log_e 2 - 2$$

14 a $f(x) = 4 - \dfrac{1}{4}x^2$

$4 - \dfrac{1}{4}x^2 = 0$

$\dfrac{1}{4}x^2 = 4$

$x^2 = 16$

$x = \pm 4$

b $g(x) = 3 - \dfrac{1}{3}x^2$

$3 - \dfrac{1}{3}x^2 = 0$

$\dfrac{1}{3}x^2 = 3$

$x^2 = 9$

$x = \pm 3$

c $\text{Area} = \int_{-4}^{4} f(x)dx - \int_{-3}^{3} g(x)dx$

$$= \int_{-4}^{4} \left(4 - \frac{1}{4}x^2\right) dx - \int_{-3}^{3} \left(3 - \frac{1}{3}x^2\right) dx$$

$$= \left[4x - \frac{x^3}{12}\right]_{-4}^{4} - \left[3x - \frac{x^3}{9}\right]_{-3}^{3}$$

$$= \left(16 - \frac{64}{12}\right) - \left(-16 + \frac{64}{12}\right) - \left[(9 - 3) - (-9 + 3)\right]$$

$$= \frac{64}{3} - (12)$$

$$= \frac{28}{3}$$

$= 9\dfrac{1}{3}$ sq. metres

15 $\text{Area} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(2\frac{1}{2} - 2x^2\right) dx - \int_{-\frac{1}{2}}^{\frac{1}{2}} 2x^2\, dx$

$$= \left[2{\cdot}5x - \frac{2x^3}{3}\right]_{-\frac{1}{2}}^{\frac{1}{2}} - \left[\frac{2x^3}{3}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \left(\frac{5}{4} - \frac{1}{12}\right) - \left(-\frac{5}{4} + \frac{1}{12}\right) - \left[\left(\frac{1}{12}\right) - \left(-\frac{1}{12}\right)\right]$$

$= 2\dfrac{1}{6}$ sq. metres

16 a $4 - \dfrac{x^2}{100} = 0$

$\dfrac{x^2}{100} = 4$

$x^2 = 400$

$x = \pm 20$

b 40 m

c $\text{Area} = \int_{-20}^{20} 5 - \left(4 - \dfrac{x^2}{100}\right) dx$

$$= \left[5x - 4x + \frac{x^3}{300}\right]_{-20}^{20}$$

$$= \left(20 + \frac{80}{3}\right) - \left(-20 - \frac{80}{3}\right)$$

$= 93\frac{1}{3}$ sq. metres

d $93\frac{1}{3} \times 9 = 840 \text{ m}^3$

17 max height of tunnel = 5 m (amplitude)

a x-intercept $= 5 \sin \frac{x\pi}{30} = 0$

$\sin \frac{x\pi}{30} = 0$

basic angle $= 0, \pi, 2\pi \ldots$

$\frac{x\pi}{30} = 0, \pi$

$x = 0, \frac{\pi \times 30}{\pi},$

$x = 0, 30$ m

length of bridge is 30 m

$$\text{Area} = \int_0^{30} \left(5 - 5\sin\frac{x\pi}{30}\right) dx$$

$$= \left[5x + \frac{5\times 30}{\pi}\cos\frac{x\pi}{30}\right]_0^{30}$$

$$= \left(150 + \frac{150}{\pi}\cos\pi\right) - \left(0 + \frac{150}{\pi}\cos 0\right)$$

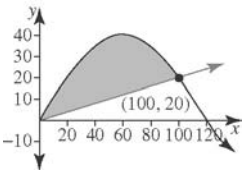
$$= 150 - \frac{150}{\pi} - \frac{150}{\pi}$$

$$= \left(150 - \frac{300}{\pi}\right) \text{m}^2$$

b $\text{Volume} = \left(150 - \frac{300}{\pi}\right) \times 200$

$= \left(30{,}000 - \frac{60000}{\pi}\right) \text{m}^3$

18 $y = 40 \sin \frac{x\pi}{120} \qquad 0 \le x \le 120$



Intersection points when

$40 \sin \frac{x\pi}{120} = \frac{x}{5}$

Use calculator to find intersection.

(100, 20)

$$A = \int_0^{100} \left(40\sin\frac{x\pi}{120} - \frac{x}{5}\right) dx$$

$$= \left[-\frac{40\times 120}{\pi}\cos\frac{\pi x}{120} - \frac{x^2}{10}\right]_0^{100}$$

$$= \left(-\frac{4800}{\pi}\cos\frac{5\pi}{6} - \frac{10000}{10}\right) - \left(-\frac{4800}{\pi}\cos 0\right)$$

$$= \frac{-4800}{\pi} \times \frac{-\sqrt{3}}{2} - 1000 + \frac{4800}{\pi}$$

$$= \frac{2400\sqrt{3}}{\pi} - 1000 + \frac{4800}{\pi}$$

$\simeq$ 1851 sq. metres

$1851 \div 12 \simeq 154$ trees

Exercise 8E — Further applications of integration — modelling and problem solving

1 $f'(x) = (2 - x)^2$

$f(x) = \frac{(2-x)^3}{-3} + c$

at $x = 0$, $\frac{(2-0)^3}{-3} + c = \frac{4}{3}$

$\frac{8}{-3} + c = \frac{4}{3}$

$c = \frac{4}{3} + \frac{8}{3}$

$c = 4$

$f(x) = \frac{-(2-x)^3}{3} + 4$

2 $\frac{dy}{dx} = 1 - 4\cos 2x$

$y = x - 2\sin 2x + c$

at $x = 0$, $0 - 2\sin 0 + c = 2$

$-2 \times 0 + c = 2$

$c = 2$

$y = x - 2\sin 2x + 2$

at $x = \frac{\pi}{12}$, $y = \frac{\pi}{12} - 2\sin\frac{\pi}{6} + 2$

$= \frac{\pi}{12} - 1 + 2$

$= \frac{\pi}{12} + 1$

3 $\frac{dy}{dx} = -0{\cdot}03(x + 1)^2 + 0{\cdot}03$

a at $x = 0$, deflection = 0 m.

b $y = \frac{-0{\cdot}03(x+1)^3}{3} + 0{\cdot}03x + c$

at $x = 0$, $y = 0$

$\frac{-0{\cdot}03(0+1)^3}{3} + 0{\cdot}03 \times 0 + c = 0$

$-0{\cdot}0 + c = 0$

$c = 0{\cdot}01$

$y = -0{\cdot}01(x + 1)^3 + 0{\cdot}03x + 0{\cdot}01$

c maximum deflection occurs when $x = 3$

$y = -0{\cdot}01(3 + 1)^3 + 0{\cdot}03 \times 3 + 0{\cdot}01$

$= -0{\cdot}54$.

deflection is 54 cm down.

4 a $\frac{dc}{dn} = 40 - 2e^{0.01n} \quad n \in [0, 200]$

for $n = 100$

$\frac{dc}{dn} = 40 - 2e^1$

$= \$34.56$

b $C = 40n - 200e^{0.01n} + c_1$

at $n = 0$, $c = 0$

$40 \times 0 - 200e^{0.01 \times 0} + c_1 = 0$

$-200 + c_1 = 0$

$c_1 = 200$

$c = 40n - 200e^{0.01n} + 200$

c at $n = 100$

$c = 40 \times 100 - 200e^{0.01 \times 100} + 200$

$= 4000 - 200e + 200$

$= \$\ 3656.34$

d i Average cost for first 100 items

$$= \frac{3656.34 - 0}{100 - 0}$$

$$= \frac{3656.34}{100}$$

$$= \$36.56$$

ii Average cost for 2nd 100 items.

$$\frac{c(200) - c(100)}{200 - 100}$$

$$c(200) = 40 \times 200 - 200e^{0.01 \times 200} + 200$$

$$= \$6722.19$$

$$\text{Average} = \frac{6722.19 - 3656.34}{200 - 100}$$

$$= \frac{3065.85}{100}$$

$$= \$30.66$$

5 $\frac{dx}{dt} = t(16 - t)$

a i at $t = 0$, $v(t) = 0$ m/s

ii at $t = 4$, $v(t) = 4(16 - 14)$

$= 48$ m/s

b i Max velocity when $v'(t) = 0$

$v = 16t - t^2$

$\frac{dv}{dt} = 16 - 2t$

for max velocity $16 - 2t = 0$

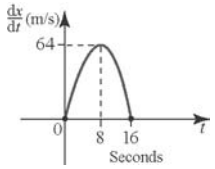
$16 = 2t$

$t = 8$ sec

ii at $t = 8$, $v(t) = 16 \times 8 - 8^2$

$= 64$ m/s

c



d $A = \int_0^{10} (16t - t^2)dt$

$$= \left[8t^2 - \frac{t^3}{3}\right]_0^{10}$$

$$= 800 - \frac{1000}{3}$$

$$= \frac{1400}{3}$$

$$= 466\frac{2}{3} \text{ m.}$$

e Area represents the distance travelled in 10 seconds.

6 $\frac{dv}{dt} = 5 + \cos\frac{\pi t}{40}$

a at $t = 40$, $\frac{dv}{dt} = 5t \cos\pi$

$= 4$ L in the 40th minute

b $V = 5t + \frac{40}{\pi}\sin\frac{\pi t}{40} + c$

at $t = 0$, $v = 0$

$5t + \frac{40}{\pi}\sin\frac{\pi t}{40} + c = 0$

$5t + c = 0$

$c = 0$

$V = 5t + \frac{40}{\pi}\sin\left(\frac{\pi t}{40}\right)$

c at $t = 40$, $V = 5 \times 40 + \frac{40}{\pi}\sin\pi$

$= 200$ L

d Average rate in the first hour

$V(60) = 5 \times 60 + \frac{40}{\pi}\sin\frac{3\pi}{2}$

$= 300 + \frac{-40}{\pi}$

$\simeq 287.27$ L

$V(0) = 5 \times 0 + \frac{40}{\pi}\sin 0$

$= 0$

$$\text{Average} = \frac{287.27 - 0}{60 - 0}$$

$= 4.79$ L/min

e Solve $5t + \frac{40}{\pi}\sin\frac{\pi t}{40} = 1600$

for t, using the graphics calculator

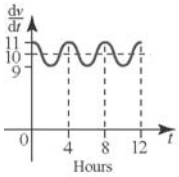
graph $Y = 5t + \frac{40}{\pi}\sin\frac{\pi t}{40}$ and $V = 1600$

and find intersection

Answer: 5 hr 20 mins

7 $\frac{dv}{dt} = 10 + \cos\frac{\pi t}{2}$

a



b Use calculator to graph $y = 10.5$

Find the intersection points.

Calculate how much time is above the line $y = 10.5$

or

$10 + \cos\frac{\pi t}{2} = 10.5$

$\cos\frac{\pi t}{2} = \frac{1}{2}$.

basic angle $= \frac{\pi}{3}$

1st and 4th quadrants

$\frac{\pi t}{2} = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 4\pi - \frac{\pi}{3}, 4\pi + \frac{\pi}{3}, 6\pi - \frac{\pi}{3}, 6\pi + \frac{\pi}{3}$

$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}, \frac{19\pi}{3}$

$t = \frac{2}{3}, \frac{10}{3}, \frac{14}{3}, \frac{22}{3}, \frac{26}{3}, \frac{34}{3}, \frac{38}{3}$

40 mins , 3 hr 20 , 4 hr 40 , 7 hr 20 , 8 hr 40 , 12 hr 40

Total time above 10.5 m is

40 mins + 1 hr 20 + 1 hr 20 + 40 min

= 4 hours.

c i 8 am to 2 pm is 6 hrs

$V = 10t + \frac{2}{\pi}\sin\frac{\pi t}{2}$

at $t = 6$, $V = 60 + \frac{2}{\pi} \sin 3\pi$

$= 60$ L

ii 3 pm to 8 pm is 5 hrs

$V = 10t + \frac{2}{\pi} \sin \frac{\pi t}{2}$

$= 50 + \frac{2}{\pi} \sin \frac{5\pi}{2}$

$= 50 + \frac{2}{\pi}$

$\simeq 50.6$ L

or find the area under the curve

$\frac{dv}{dt} = 10 + \cos \frac{\pi t}{40}$

Area $= \int_0^6 10 + \cos \frac{\pi t}{40} = 60$ L

Area $= \int_7^{12} 10 + \cos \frac{\pi t}{40} = 50.6$L

8 a Area $= \int_{-25}^{25} \left(20 - 0.024x^2\right) dx$

$= \left[20x - 0.008x^3\right]_{-25}^{25}$

$= 375 + 375$

$= 750$ m^2.

Volume $= 750 \times 75$

$= 56{,}250$ m^3.

b $56{,}250 \div 11250 = 5$ airconditioners

9 Turning point of parabola is at origin.

Two points on graph or (1.5, 2) and (–1.5, 2). Use $y = a(x + h)^2 + k$ to find rule.

$y = a(x + 0)^2 + 0$

$= 1.5^2 a$

$a = \frac{2}{1.5^2}$

$= \frac{8}{9}$.

Rule for parabola $y = \frac{8}{9}x^2$

Look at right side of parabola and find the area from $x = 0$ to $x = 1.5$

$\int_0^{1.5} \left(2 - \frac{8x^2}{9}\right) dx$

$= \left[2x - \frac{8x^3}{27}\right]_0^{1.5}$

$= 3 – 1$

$= 2$ sq. units.

At point $p(a, b)$, $b = \frac{8a^2}{9}$

Area from $x = 0$ to $x = a$ is

$\int_0^a \left(b - \frac{8x^2}{9}\right) dx = \int_0^a \left(\frac{8a^2}{9} - \frac{8x^2}{9}\right) dx$

$= \left[\frac{8a^2 x}{9} - \frac{8x^3}{29}\right]_0^a$

$= \frac{8a^3}{9} - \frac{8a^3}{27}$

$= \frac{16a^3}{27}$

This area needs to be half of the original

i.e. $A = 1$

$\frac{16a^3}{27} = 1$

$a^3 = \frac{27}{16}$

$a = \sqrt[3]{\frac{27}{16}}$

$\simeq 1.19$.

$b = \frac{8}{9} \times 1.19^2$

$\simeq 1.26$

Height of water when channel is half full is 1.26 m

10 Let $P = (a, a^3)$

Area of rectangle $= a \times a^3 = a^4$

Area under curve $= \int_0^a x^3 \, dx$

$= \left[\frac{x^4}{4}\right]_0^a$

$= \frac{a^4}{4} = \frac{1}{4}a^4$

$= \frac{1}{4}$ of area of rectangle

11 a Turning point = (0, 6)

(4, 0), (–4, 0), inverted parabola.

$y = a(x + h)^2 + k$.

$y = a(x)^2 + 6$

$0 = a(4)^2 + 6$

$-6 = a \times 16$

$\frac{-6}{16} = a$

$a = \frac{-3}{8}$

Rule: $y = \frac{-3}{8}x^2 + 6$

b $A = \int_{-4}^{4} \left(\frac{-3}{8}x^2 + 6\right) dx$

$= \left[\frac{-3x^3}{24} + 6x\right]_{-4}^{4}$

$= \left(\frac{-3 \times 64}{24} + 24\right) - \left(\frac{-3 \times 64}{24} + 6 \times -4\right)$

$= 16 + 16$

$= 32$ m^2

Area of rectangle $= 7 \times 10 = 70$ m^2

Area shaded $= 70 – 32$

$= 38$ m^2

c Volume $= 38 \times 10$

$= 380$ m^3

12 a $f(x) = e^x – 1$

$g(x) = -2e^{-x} + 2$

Intersect when

$e^x – 1 = -2e^{-x} + 2$

$e^x – 1 + 2e^{-x} – 2 = 0$

$e^x + \frac{2}{e^x} – 3 = 0$

$e^{2x} + 2 – 3e^x = 0$

Let $a = e^x$

$a^2 + 2 – 3a = 0$

$(a – 2)(a – 1) = 0$

$a = 1$ or 2

if $a = 1$, $e^x = 1$

$x = 0$

if $a = 2$, $e^x = 2$

$x = \log_e 2$

$\Rightarrow y = e^{\log_e 2} - 1 = 1$

$\Rightarrow$ Point B is $(\log_e 2, 1)$

b $A = \int_0^{\log_e 2} \left(-2e^{-x} + 2 - e^x + 1\right) dx$

$= \left[2e^{-x} + 2x - e^x + x\right]_0^{\log_e 2}$

$= \left[e^{\frac{2}{x}} + 3x - e^x\right]_0^{\log_e 2}$

$= \left(e^{\frac{2}{\log_e 2}} + 3\log_e 2 - e^{\log_e 2}\right) - \left(\frac{2}{e^0} + 0 - e^0\right)$

$= 1 + 3\log_e 2 - 2 - 2 + 1$

$= 3\log_e 2 - 2$ sq. units

c Area under $f(x)$

$= \int_0^{\log_e 2} \left(e^x - 1\right) dx$

$= \left[e^x - x\right]_0^{\log_e 2}$

$= 2 - \log_e 2 - 1$

$= 1 - \log_e 2$. sq. units

Area under g(x)

$= \int_0^{\log e^2} \left(-2e^{-x} + 2\right) dx$

$= \left[2e^{-x} + 2x\right]_0^{\log_e 2}$

$= 1 + 2\log_e 2 + 2$

$= 2\log_e 2 - 1$ sq. units

Sum of two areas

$= 1 - \log_e 2 + 2\log_e 2 - 1$

$= \log_e 2$ sq. units

Area of rectangle OCBA

$= \log_e 2 \times 1$

$= \log_e 2$ sq. units

13 a $p(t) = 12\log_e (t + 1)$

at $t = 0$, $p(t) = 0$

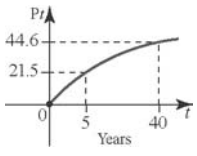
at $t = 5$, $p(t) = 12\log_e 6$

$\simeq 21.5 \ \frac{\text{kangaroos}}{\text{year}}$

at $t = 40$, $p(t) = 12\log_e 41$

$\simeq 44.6 \ \frac{\text{kangaroos}}{\text{year}}$

b



c $y = 12\log_e (t + 1)$

$t = 12\log_e (y + 1)$

$\frac{t}{12} = \log_e (y + 1)$

$y + 1 = e^{\frac{t}{12}}$

$y = e^{\frac{t}{12}} - 1$

$p^{-1}(t) = e^{\frac{t}{12}} - 1$

d Can't find area $\int_0^{40} 12\log_e (t+1) dt$

but can find area $\int_0^{44.6} \left(e^{\frac{t}{12}} - 1\right) dt$

if $t = 40$, $y = 44.6$

Area of rectangle $= 40 \times 44.6$

$= 1784$

Area under inverse

$= \int_0^{44.6} \left(e^{\frac{t}{12}} - 1\right) dt$

$= \left[12e^{\frac{t}{12}} - t\right]_0^{44.6}$

$\simeq (449) - (12)$

$\simeq 437$

$1784 - 437 = 1347$

e This area represents the population of kangaroos on 1 January 1995 (40 years after start).

Investigation — Concrete chute

$f(x) = 0.12x^2 - 12 - 10 \le x \le 10$

a Area $= -\int_{-10}^{10} \left(0.12x^2 - 12\right) dx$

$= -\left[0.04x^3 - 12x\right]_{-10}^{10}$

$= -\left[(40 - 120) - (-40 + 120)\right]$

$= -(-80 - 80)$

$= 160$ cm^2.

b Area of rectangle shape

$= 5 \times 20$

$= 100$ cm^2

Total area $= 160 + 100$

$= 260$ cm^2

$260 \text{ cm}^2 \times \frac{1 \text{ m}^2}{10000 \text{ cm}^2} = 0.026 \text{ m}^2$

0.026×1.6 m/s $\times$ 60 s/min

$= 2.496$ m^3 every minute

c 12 m^3 $\div$ 2.496 m^3/min

$= 4.8077$ min

$= 4$ min 48 sec

Chapter review

1 $\int_0^4 \left(3\sqrt{x} - x\right) dx = \int_0^4 \left(3x^{\frac{1}{2}} - x\right) dx$

$= \left[2x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^4$

$= 16 - 8$

$= 8$

Answer is **B**

2 $\int_{-2}^{2} \left(4e^{2x} - 2e^{-2x}\right) dx$

$= \left[2e^{2x} + e^{-2x}\right]_{-2}^{2}$

$= (2e^4 + e^{-4}) - (2e^{-4} + e^4)$

$= e^4 - e^{-4}$

Answer is **E**

3 $\int_0^{\pi} -2\cos\frac{x}{3} dx$

$= \left[-6\sin\frac{x}{3}\right]_0^{\pi}$

$= -6\sin\frac{\pi}{3} - -6\sin 0$

$= -6 \times \frac{\sqrt{3}}{2}$

$= -3\sqrt{3}$

Answer is **C**

4 a $\int_{-1}^{0} \frac{9}{(2x+3)^4}\,dx = \int_{-1}^{0} 9(2x+3)^{-4}\,dx$

$= \left[\frac{9(2x+3)^{-3}}{2\times -3}\right]_{-1}^{0}$

$= \left[\frac{-3}{2(2x+3)^3}\right]_{-1}^{0}$

$= -\frac{1}{18} + \frac{3}{2}$

$= \frac{13}{9}$

$= 1\frac{4}{9}$ sq. units

b $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos 2x\,dx$

$= \left[\frac{1}{2}\sin 2x\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$

$= \frac{1}{2}\sin\frac{4\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3}$

$= \frac{1}{2} \times \frac{-\sqrt{3}}{2} - \frac{1}{2} \times \frac{\sqrt{3}}{2}$

$= -\frac{\sqrt{3}}{2}$

5 $\int_0^k (4x-5)\,dx = -2$

$\left[2x^2 - 5x\right]_0^k = -2$

$2k^2 - 5k = -2$

$2k^2 - 5k + 2 = 0$

$(2k-1)(k-2) = 0$

$k = \frac{1}{2}$ or 2

6 $\int_2^4 (x-2)^3\,dx$

$= \left[\frac{(x-2)^4}{4}\right]_2^4$

$= 4 - 0$

$= 4$ sq. units

Answer is **D**

7 $\int_{-2}^{2} \left(-1-3x^3\right)dx$

$= \left[-x - x^3\right]_{-2}^{2}$

$= (-2 - 8) - (2 + 8)$

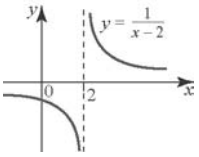
$= -10 - 10$

$= -20$

Area is 20 sq. units

Answer is **A**

8 a



b $\int_3^6 \frac{1}{x-2}\,dx$

$= \left[\log_e(x-2)\right]_3^6$

$= \log_e 4 - \log_e 1$

$= \log_e 4$

9 $A = \int_{-2}^{0} x(x+2)(x-3)\,dx + -\int_0^3 x(x+2)(x-3)\,dx$

$= \int_{-2}^{0} \left(x^3 - x^2 - 6x\right)dx - \int_0^3 \left(x^3 - x^2 - 6x\right)dx$

$= \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_0^3$

$= -\left(4 + \frac{8}{3} - 12\right) - \left[\left(\frac{81}{4} - 9 - 27\right) - 0\right]$

$= 8 - \frac{8}{3} - \frac{81}{4} + 36$

$= \frac{253}{12}$

$= 21\frac{1}{12}$ sq. units

Answer is **B**

10 $f(x) = e^x - 1$

e^x graph translated down 1

asymptote at $y = -1$

y-intercept at 0

Answer is **C**

11 $\int_0^2 (e^x - 1)dx = \left[e^x - x\right]_0^2$

$= (e^2 - 2) - (1)$

$= e^2 - 3$

Answer is **E**

12 Area of rectangle $= e^2 - 1 \times 2$

$= 2e^2 - 2$

Area under curve $= e^2 - 3$

Area bounded by curve and y-axis is

$2e^2 - 2 - e^2 + 3 \quad = e^2 + 1$

Answer is **D**

13 $A = \int_{-6}^{4} (4-x)(6+x)dx$

$= \int_{-6}^{4} (24 - 2x - x^2)dx$

$= \left[24x - x^2 - \frac{x^3}{3}\right]_{-6}^{4}$

$= \left(96 - 16 - \frac{64}{3}\right) - \left(-144 - 36 + \frac{216}{3}\right)$

$= \frac{176}{3} - -108$

$= 166\frac{2}{3}$ sq. units

14 $2x + 3 = x^2$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3$ and $x = -1$

Answer is **B**

15 $\int_{-1}^{3} (2x + 3 - x^2)dx$

$= \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3}$

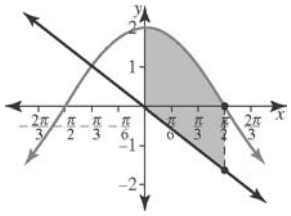
$= (9 + 9 - 9) - (1 - 3 + \frac{1}{3})$

$= 9 + \frac{5}{3}$

$= 10\frac{2}{3}$ sq. units

Answer is **A**

16



$\int_0^{\frac{\pi}{2}} (2\cos x + x)\,dx$

$= \left[2\sin x + \frac{x^2}{2}\right]_0^{\frac{\pi}{2}}$

$= 2\sin\frac{\pi}{2} + \frac{\pi^2}{8} - 2\sin 0 - 0$

$= 2 + \frac{\pi^2}{8}$ sq. units

$\simeq 3.23$ sq. units

17 a $y = e^{\frac{x}{2}}$

at $x = 2$, $y = e$

$A = (2, e)$

b $\frac{dy}{dx} = \frac{1}{2}e^{\frac{x}{2}}$

at $x = 2$, $m = \frac{1}{2}e$

m of normal $= \frac{-2}{e}$

Equation of normal at A

$y - e = \frac{-2}{e}(x - 2)$

$y = \frac{-2x}{e} + \frac{4}{e} + e$

$y = -2e^{-1}x + 4e^{-1} + e$

c y-intercept of normal is $\left(0, e + \frac{4}{e}\right)$

d y-intercept of $y = e^{\frac{x}{2}}$ is 1 or (0, 1)

e $\int_0^2 \frac{-2x}{e} + \frac{4}{e} + e - e^{\frac{x}{2}}\,dx$

$= \left[\frac{-x^2}{e} + \frac{4x}{e} + ex - 2e^{\frac{x}{2}}\right]_0^2$

$= \left(\frac{-4}{e} + \frac{8}{e} + 2e - 2e\right) - (0 + 0 + 0 - 2)$

$= \frac{4}{e} + 2$ sq. units

18 a $f(x) = x\log_e x$

$f'(x) = \frac{1}{x} \times x + \log_e x$

$= 1 + \log_e x$

b $\int (1 + \log_e x)\,dx = x\log_e x$

$\int 1\,dx + \int \log_e x\,dx = x\log_e x$

$x + \int \log_e x\,dx = x\log_e x$

$\int \log_e x\,dx = x\log_e x - x$

c at $x = e^{-2}$, $f(x) = \log_e e^{-2}$

$= -2$

Height of platform $= (-2) + 1$

$= 3$ m

d Area under curve

$= \int_{e^{-2}}^{2} (\log_e x + 2)\,dx$

move curve up 2, to find area between curve and x-axis

$= \left[x\log_e x - x + 2x\right]_{e^{-2}}^{e}$

$= \left[x\log_e x + x\right]_{e^{-2}}^{e}$

$= \left(e^{-2}\log_e + e\right) - \left(e^{-2}\log_e e^{-2} + e^{-2}\right)$

$= (e + e) - (-2e^{-2} + e^{-2})$

$= 2e - -e^{-2}$

$= 2e + e^{-2}$ sq. units

Area of side column $= 3 \times e^{-2} = \frac{3}{e^2}$

Area of rectangle $= 3 \times e = 3e$

Shaded area $= 3e - (2e + e^{-2})$

$= 3e - e^{-2}$ m^2

$= 2.58$ m^2

e Volume $= 20(e - e^{-2})$ m^3

≈ 51.66 m^3

Modelling and problem solving

1 a Let $y = a(x - 3)(x + 3)$

$a(x^2 - 9)$

Using (0, 3),

$3 = a(0^2 - 9)$

$a = \frac{-1}{3}$

$\therefore y = \frac{-1}{3}(x^2 - 9)$

b Area ($\frac{1}{2}$ of parabola)

Area $= \int_0^3 \frac{-1}{3}(x^2 - 9)\,dx$

$= \frac{-1}{3}\left[\frac{x^3}{3} - 9x\right]_0^3$

$= \frac{-1}{3}[(9 - 27) - (0)]$

$= 6$

Area of parabola = Area of window $= 2 \times 6 = 12$ m^2

c $\frac{2}{3}$ base of arch $\times$ height

$= \frac{2}{3} \times 6 \times 3$

$= 12$ m^2

Therefore the area of the window is equal to $\frac{2}{3}$ base of arch $\times$ height.

d Base $= 8$

Height $= 4.5$

Area $= \frac{2}{3} \times 8 \times 4.5$

$= 24$ m^2

2 a $\frac{dA}{dt} = 2t + 6t^2 - \frac{1}{4}t^3$

$A = \int 2t + 6t^2 - \frac{1}{4}t^3 \, dt$

$A = t^2 + 2t^3 - \frac{t^4}{16} + c$

$t = 0, A = 10 \text{ cm}^2$

$A = t^2 + 2t^3 - \frac{t^4}{16} + 10$

$t = ?, A = 0.6 \text{ m}^2 = 6000 \text{ cm}^2$

$t^2 + 2t^3 - \frac{t^4}{16} + 10 = 6000$

$t^4 - 32t^3 - 16t^2 + 95840 = 0$

Use a calculator to solve for t.

$t = 19.11$ and 28.36

It will take 19 weeks to cover 0.6 m^2.

b Maximum area will occur when $\frac{dA}{dt} = 0$

$2t + 6t^2 - \frac{1}{4}t^3 = 0$

$8t + 24t^2 - t^3 = 0$

$t(t^2 - 24t - 8) = 0$

$t = 0 \text{ or } t = \frac{24 \pm \sqrt{24^2 - 4(1)(8)}}{2}$

$t = 23.66$ or 0.3380

For maximum area, $t = 23.66$

$A = (23.66)^2 + 2(23.66)^3 - \frac{(23.66)^4}{16} + 10$

$A = 7473.63 \text{ cm}^2$

$A = 0.75 \text{ m}^2$

3 $\frac{dV}{dt} = 300e^{0.3t}$

$V = \int 300e^{0.3t} \, dt$

$= 300 \times \frac{1}{0.3}e^{0.3t} + c$

$= 1000e^{0.3t} + c$

$t = 0, V = \$1000$

$c = 0$

$V = 1000e^{0.3t}$

When $t = 6, V = 1000e^{0.3(6)}$

$= 6049.65$

After 6 years the value is \$6050 (to the nearest dollar).

4 $\frac{dN}{dt} = Ae^{-0.2t}$

a $N = \int Ae^{-0.2t} \, dt$

$= \frac{A}{-0.2}e^{-0.2t} + c$

$c = 0$

$N = \frac{A}{-0.2}e^{-0.2t}$

b $t = 0, A = -2000$

$N = \frac{-2000}{-0.2}e^{-0.2(0)}$

$N = 10\,000$

c $t = 2$

$N = \frac{-2000}{-0.2}e^{-0.2(2)}$

$N = 6703.2$

$N = 6703$ people

d $t = 10$

$N = \frac{-2000}{-0.2}e^{-0.2(10)}$

$N = 1353.35$

$N = 1353$ people

e The number of cases is modelled by an exponential function. N will approach but never actually reach zero.

f $N = 100$

$100 = 10000e^{-0.2t}$

$0.01 = e^{-0.2t}$

$\log_e(0.01) = -0.2t$

$t = 23.026$

The number of cases will be 100 after approximately 23 years.